Anneli Lax New Mathematical Library, #44

Geometric Transformations IV

CIRCULAR TRANSFORMATIONS

I. M. Yaglom

Translated by
A. Shenitzer

Mathematical Association of America

Geometric Transformations IV

Circular Transformations

Library of Congress Catalog Card Number 2009933072

Print edition ISBN 978-0-88385-648-2

Electronic edition ISBN 978-0-88385-958-2

Printed in the United States of America

Current Printing (last digit):
10 9 8 7 6 5 4 3 2 1

Geometric Transformations IV

Circular Transformations

I. M. Yaglom

Translated by

A. Shenitzer

Published and Distributed by
The Mathematical Association of America

ANNELI LAX NEW MATHEMATICAL LIBRARY

PUBLISHED BY

THE MATHEMATICAL ASSOCIATION OF AMERICA

ANNELI LAX NEW MATHEMATICAL LIBRARY

1. Numbers: Rational and Irrational *by Ivan Niven*
2. What is Calculus About? *by W. W. Sawyer*
3. An Introduction to Inequalities *by E. F. Beckenbach and R. Bellman*
4. Geometric Inequalities *by N. D. Kazarinoff*
5. The Contest Problem Book I Annual High School Mathematics Examinations 1950–1960. Compiled and with solutions *by Charles T. Salkind*
6. The Lore of Large Numbers *by P. J. Davis*
7. Uses of Infinity *by Leo Zippin*
8. Geometric Transformations I *by I. M. Yaglom, translated by A. Shields*
9. Continued Fractions *by Carl D. Olds*
10. Replaced by NML-34
11. }
12. } Hungarian Problem Books I and II, Based on the Eötvös Competitions 1894–1905 and 1906–1928, *translated by E. Rapaport*
13. Episodes from the Early History of Mathematics *by A. Aaboe*
14. Groups and Their Graphs *by E. Grossman and W. Magnus*
15. The Mathematics of Choice *by Ivan Niven*
16. From Pythagoras to Einstein *by K. O. Friedrichs*
17. The Contest Problem Book II Annual High School Mathematics Examinations 1961–1965. Compiled and with solutions *by Charles T. Salkind*
18. First Concepts of Topology *by W. G. Chinn and N. E. Steenrod*
19. Geometry Revisited *by H. S. M. Coxeter and S. L. Greitzer*
20. Invitation to Number Theory *by Oystein Ore*
21. Geometric Transformations II *by I. M. Yaglom, translated by A. Shields*
22. Elementary Cryptanalysis *by Abraham Sinkov, revised and updated by Todd Feil*
23. Ingenuity in Mathematics *by Ross Honsberger*
24. Geometric Transformations III *by I. M. Yaglom, translated by A. Shenitzer*
25. The Contest Problem Book III Annual High School Mathematics Examinations 1966–1972. Compiled and with solutions *by C. T. Salkind and J. M. Earl*
26. Mathematical Methods in Science *by George Pólya*
27. International Mathematical Olympiads—1959–1977. Compiled and with solutions *by S. L. Greitzer*
28. The Mathematics of Games and Gambling, Second Edition *by Edward W. Packel*
29. The Contest Problem Book IV Annual High School Mathematics Examinations 1973–1982. Compiled and with solutions *by R. A. Artino, A. M. Gaglione, and N. Shell*
30. The Role of Mathematics in Science *by M. M. Schiffer and L. Bowden*
31. International Mathematical Olympiads 1978–1985 and forty supplementary problems. Compiled and with solutions *by Murray S. Klamkin*
32. Riddles of the Sphinx *by Martin Gardner*
33. U.S.A. Mathematical Olympiads 1972–1986. Compiled and with solutions *by Murray S. Klamkin*
34. Graphs and Their Uses *by Oystein Ore.* Revised and updated *by Robin J. Wilson*
35. Exploring Mathematics with Your Computer *by Arthur Engel*

36. Game Theory and Strategy *by Philip D. Straffin, Jr.*
37. Episodes in Nineteenth and Twentieth Century Euclidean Geometry *by Ross Honsberger*
38. The Contest Problem Book V American High School Mathematics Examinations and American Invitational Mathematics Examinations 1983–1988. Compiled and augmented *by George Berzsenyi and Stephen B. Maurer*
39. Over and Over Again *by Gengzhe Chang and Thomas W. Sederberg*
40. The Contest Problem Book VI American High School Mathematics Examinations 1989–1994. Compiled and augmented *by Leo J. Schneider*
41. The Geometry of Numbers *by C. D. Olds, Anneli Lax, and Giuliana P. Davidoff*
42. Hungarian Problem Book III, Based on the Eötvös Competitions 1929–1943, *translated by Andy Liu*
43. Mathematical Miniatures *by Svetoslav Savchev and Titu Andreescu*
44. Geometric Transformations IV *by I. M. Yaglom, translated by A. Shenitzer*

Other titles in preparation.

Books may be ordered from:
MAA Service Center
P. O. Box 91112
Washington, DC 20090-1112
1-800-331-1622 fax: 301-206-9789

Contents

1

Reflection in a circle (inversion)

To construct the image A' of a point A by reflection in a line l we usually proceed as follows. We draw two circles with centers on l passing through A. The required point A' is the second point of intersection of the two circles (Figure 1). We say of A' that it is *symmetric to A with respect to l*.

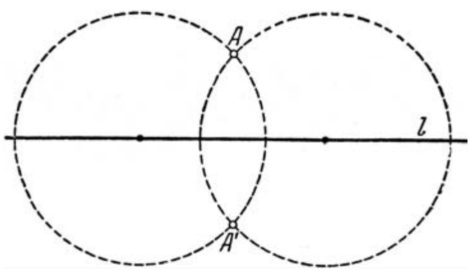


FIGURE 1

Here we are making use of the fact that all circles with centers on a line l passing through a point A pass also through the point A' symmetric to A with respect to l (Figure 2). This fact can be used as a definition of a reflection in a line: *Points A and A' are said to be symmetric with respect to a line l if every circle with center on l passing through A passes also through A'*. It is clear that this definition is equivalent to the one in NML 8, p. 41.

In this section we consider a *reflection in a circle*. This transformation is analogous in many respects to a reflection in a line and is often useful in the solution of geometric problems.

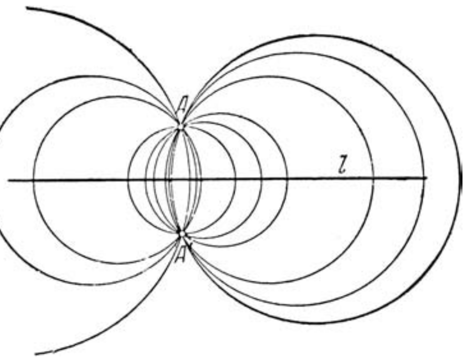


FIGURE 2

Our starting point is the definition of a reflection in a line just given. We modify this definition slightly.

In this book we will frequently speak of an angle between two circles or an angle between a circle and a line. Draw the tangents to two circles at a point of their intersection. It is natural to call the angle between these tangents the *angle between the two circles* at that point (Figure 3a). This definition implies that the angle between two circles is equal to the angle between the radii to the point of tangency (or to the adjacent angle; the angle between two circles, as well as the angle between two lines, is not uniquely defined: we can say that it is equal to α or to $180° - \alpha$). Similarly, if a line l intersects a circle S in a point, then we call the angle between the line and the tangent to the circle at that point the *angle between the line and the circle* (Figure 3b).[1]

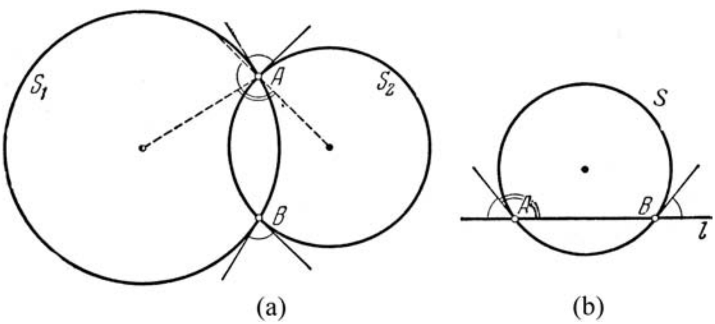


FIGURE 3

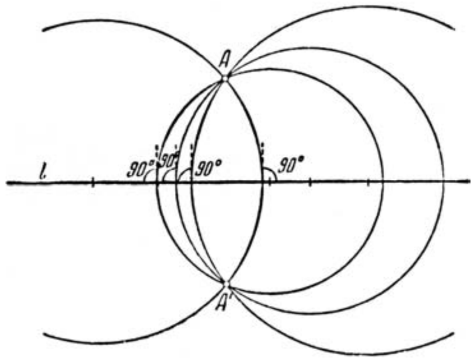


FIGURE 4

The circles whose centers lie on a line l (and only these circles) are perpendicular to l (Figure 4). This justifies the following definition of a reflection in a line: *Points A and A' are symmetric with respect to a line l if every circle passing through A and perpendicular to l passes also through A'.*

We now prove the following theorem.

Theorem 1. *All circles passing through a point A and perpendicular to a given circle* Σ (not passing through A) *pass also through a point A' different from A.*

Proof. Let S be a circle passing through A and perpendicular to the circle Σ (Figure 5). Draw a line through A and the center O of Σ and join O to the point B of intersection of Σ and S. Since S is perpendicular to Σ, OB is tangent to S. Let A' be the second point of intersection of the line

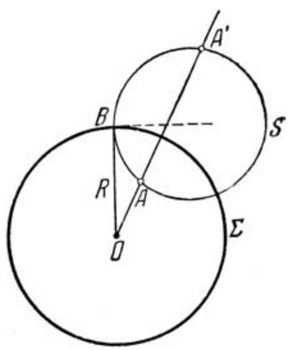


FIGURE 5

OA and the circle S. In view of a well-known property of a tangent to a circle, we have

$$OA \cdot OA' = OB^2,$$

or

$$OA' = \frac{R^2}{OA},$$

where R is the radius of Σ. This shows that A', the point of intersection of OA and S, does not depend on the choice of S. Hence all circles S perpendicular to Σ and passing through A intersect the line OA in the same point A' (Figure 6), which is what we wished to prove.

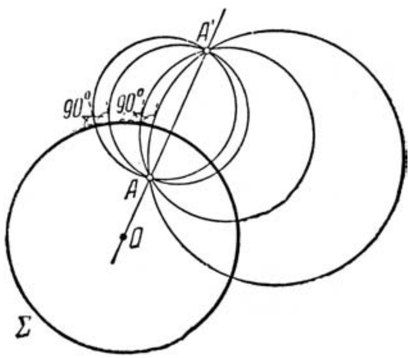


FIGURE 6

Now the following is seen to be a natural definition: *A point A is said to be symmetric to a point A' with respect to a circle Σ if every circle passing through A and perpendicular to Σ passes also through A'.*[2] It is clear that if A' is symmetric to A with respect to Σ, then A is symmetric to A' with respect to Σ (see Figure 6). This allows us to speak of points mutually symmetric with respect to a circle. The totality of points symmetric to the points of a figure F with respect to a circle Σ forms a figure F' symmetric to F with respect to Σ (Figure 7). If A' is symmetric to A with respect to Σ, then we also say that A' is obtained from A by reflection in Σ.

It is reasonable to regard a reflection in a line as a limiting case of a reflection in a circle. The reason for this is that we may think of a line as a "circle of infinite radius." We will see in the sequel that inclusion of lines in the class of circles simplifies many arguments connected with reflections in circles.

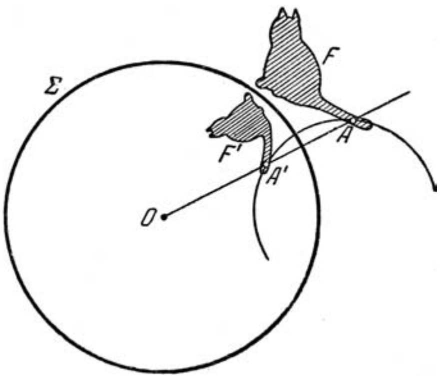


FIGURE 7

A reflection in a circle is also called an inversion. Then the circle one reflects in is called the circle of inversion, its center is called the center of inversion, and $k = R^2$, the square of its radius, is called the power of the inversion. The term "inversion" has less intuitive appeal than the term "reflection in a circle" but it has the virtue of brevity. Hence its popularity. We will routinely use this term in the sequel.

It is clear that an inversion can also be defined as follows: *An inversion with power k and center O is a mapping that takes a point A in the plane to a point A' on the ray OA such that*

$$OA' = \frac{k}{OA} \qquad (*)$$

(Figure 8a).[3] Obviously, this definition is equivalent to the one given earlier (see the proof of Theorem 1). While it is less geometric than the previous one, this definition has the virtue of simplicity.[4]

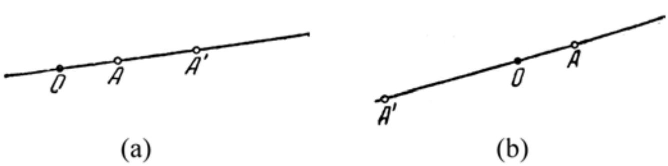


FIGURE 8

It is sometimes convenient to consider a transformation that takes a point A to a point A' such that A and A' are collinear with a fixed point O, lie on either side of O, and $OA' = k/OA$ (Figure 8b). We call such a mapping an

inversion with center O and negative power $-k$.[5] It is clear that an inversion with center O and negative power $-k$ is equivalent to an inversion with center O and positive power k (a reflection in the circle S with center O and radius $\sqrt{k}$) followed by a reflection in O.

An inversion with negative power can be defined geometrically in a manner analogous to the first definition of an inversion with positive power (a reflection in a circle). In this connection the following theorem plays a key role.

Theorem 1′. *All circles passing through a given point A and intersecting a given circle Σ in two diametrically opposite points pass also through a point A' different from A.*

In fact, let B_1 and B_2 be the endpoints of the diameter of a circle Σ in which a circle S passing through the given point A intersects Σ (Figure 9a). Draw a line through A and the center O of Σ and denote by A' its second point of intersection with the circle S. Let MN be the diameter of S passing through O. Since the center of S is equidistant from B_1 and B_2 and O is the midpoint of the segment B_1B_2, it follows that $MN \perp B_1B_2$. This implies that OB_1 is an altitude of the right triangle MB_1N. By a well-known theorem $OB_1^2 = MO \cdot ON$. On the other hand, by the property of chords in a circle, $MO \cdot ON = A'O \cdot OA$. Hence

$$A'O \cdot OA = OB_1^2,$$

that is,

$$A'O = \frac{R^2}{OA}, \qquad (**)$$

where R is the radius of Σ. This implies that $A'O$ is independent of the choice of the circle S. Hence all circles passing through A and intersecting Σ in diametrically opposite points pass through the same point A'.

The relation $(**)$ shows that *if every circle passing through a point A and intersecting a circle Σ with center O and radius R in diametrically opposite points passes also through another point A', then A' is obtained from A by an inversion with* (negative) *power $-R^2$ and center O.*

We now list some basic properties of inversion.

A. *Under an inversion with center O and power k the circle Σ with center O and radius $\sqrt{|k|}$* (if k is positive, then Σ is the circle of inversion) *goes over to itself, the points in its interior* (other than its center) *go over to points in its exterior, and conversely.*

The center O of inversion is the only point in the plane that does not go over to any point in the plane (nor is it the image of any point in the plane).

Property A of inversion follows immediately from the basic formula $(*)$: If $OA = \sqrt{|k|}$ (the point A lies on the circle of inversion Σ), then $OA' = |k/OA| = \sqrt{|k|}$, that is, A' is also on Σ; if $OA < \sqrt{|k|}$ (the point

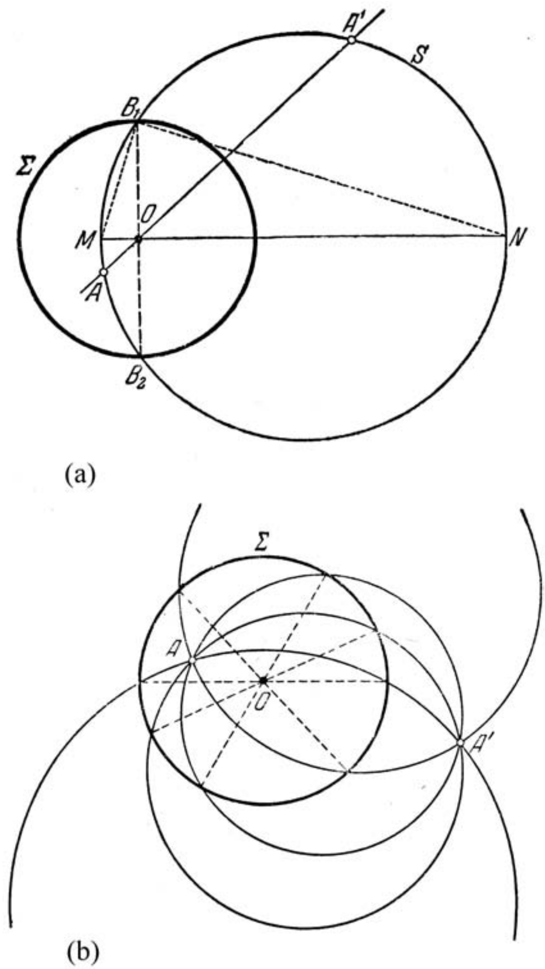


FIGURE 9

A lies in the interior of Σ), then $OA' = |k/OA| > \sqrt{|k|}$, that is, A' lies in the exterior of Σ; if $OA > \sqrt{|k|}$, then $OA' = |k/OA| < \sqrt{|k|}$.

We note that if the power of an inversion is positive (the inversion is a reflection in a circle Σ), then every point of Σ goes over to itself (is a fixed point of the inversion), and if the power is negative, then every point of Σ goes over to the diametrically opposite point of Σ (an inversion with negative power has no fixed points).

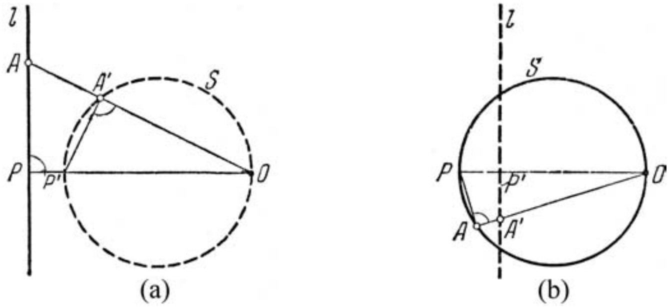


FIGURE 10

B$_1$. *An inversion maps a line l passing through the center of inversion O to itself.*

The proof of this assertion is an immediate consequence of the fact that the image A' of a point A under inversion lies on the line OA.

B$_2$. *An inversion maps a line l not passing through the center of inversion O on a circle passing through O.*

Let P be the point of intersection of l and the perpendicular from O to l and let P' be its image under inversion (see Figure 10a which depicts the case of an inversion with positive power; the proof is virtually the same if the power is negative). The second definition of an inversion implies that

$$OP' = \frac{k}{OP},$$

where k is the power of the inversion. Let A be a point on l and let A' be its image under inversion. Then

$$OP \cdot OP' = OA \cdot OA' = k,$$

so that

$$\frac{OP}{OA} = \frac{OA'}{OP'}.$$

The latter implies the similarity of the triangles OPA and $OA'P'$ (for they share an angle and the ratios of its sides in the two triangles are equal). Hence $\angle OA'P' = \angle OPA = 90°$, that is, A' lies on the circle S whose diameter is the segment OP'.

B$_3$. *An inversion maps a circle S passing through the center of inversion O on a line l not passing through O.*

Let P be the second endpoint of the diameter of S passing through the center O and let P' be its image under our inversion. Let A be a point

on S with image A' (Figure 10b). As before, we prove that the triangles $OP'A'$ and OAP are similar (for $OP \cdot OP' = OA \cdot OA' = k$), so that $\angle OP'A' = \angle OAP = 90°$, that is, A' lies on the line through P' perpendicular to OP.

B$_4$. *An inversion maps a circle S not passing through the center of inversion O on a circle S'* (not passing through O).

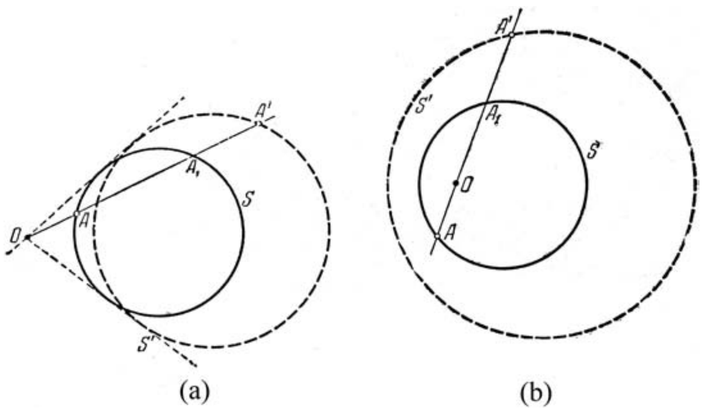


FIGURE 11

Let A be a point on S with image A' under our inversion and let A_1 be the second point of intersection of the line OA and S (Figures 11a and 11b). We have

$$OA' = \frac{k}{OA},$$

where k is the power of the inversion. On the other hand, in view of a well-known property of circles,

$$OA \cdot OA_1 = k_1, \qquad OA = \frac{k_1}{OA_1},$$

where k_1 is independent of A (if O is exterior to S, then k_1 is the square of the tangent from O to S). The last two formulas imply that

$$OA' = \frac{k}{k_1} \cdot OA_1,$$

that is, A' is centrally similar to A_1 under the similarity with center O and coefficient k/k_1, and hence is on the circle S' that is centrally similar to S under this similarity.[6] This proves our assertion.

We note that our inversion does not map the center of S on the center of S'.

We give another proof of property B_4 of inversion which is closer to the proofs of properties B_2 and B_3.

Let Q be the center of S, M and N the points of intersection of the circle S with the line OQ (O is the center of inversion), A any point on S, and M', N', A' the images under inversion of M, N, A (Figure 12). Just as in proving properties B_2 and B_3, we show that $\angle OAM = \angle OM'A'$ and $\angle OAN = \angle ON'A'$. Obviously, $\angle MAN = \angle OAN - \angle OAM$ and $\angle M'A'N' = \angle ON'A' - \angle OM'A'$. Hence

$$\angle M'A'N' = \angle MAN = 90^\circ,$$

which means that A' lies on the circle S' with diameter $M'N'$.

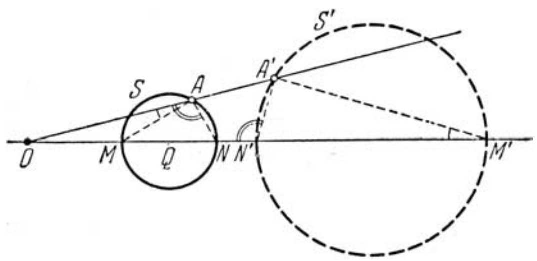


FIGURE 12

The following proposition combines properties B_1 through B_4 of inversion.

B. *An inversion maps a circle or a line on a circle or a line.*

We note the following consequence of property B_3. *Given a circle, there is always an inversion that maps it on a line.* In fact, all we need do is choose any of its points as the center of inversion.

It is easy to see that there is always an inversion which maps one of two given circles on the other. If the given circles are incongruent, then this can be done in two ways by taking as the respective centers of inversion the exterior and interior centers of similarity of these circles (see the proof of property B_4). If the two circles are congruent, then there is just one inversion which maps one of them on the other (two congruent circles have just one center of similarity; see NML 21, p. 14). But in that case we can map the two circles on each other by reflection in their axis of symmetry, and this reflection can be viewed as a limiting case of inversion (see p. 5).

Similarly, there are two inversions which map a circle and a line not tangent to it on each other. The centers of the two inversions are the endpoints of the diameter of the circle perpendicular to the line (see the proofs of properties B_2 and B_3). If the

circle and the line are tangent to one another, then there is only one inversion which maps the two on each other.

Two lines cannot be mapped on each other by an inversion but can be mapped on each other by a reflection in a line. If the two lines are not parallel, then there are two such reflections (their axes are the two perpendicular bisectors of the angles formed by the lines), and if they are parallel, then there is just one such reflection.

C. *Inversion preserves the angle between two circles* (or between a circle and a line or between two lines).

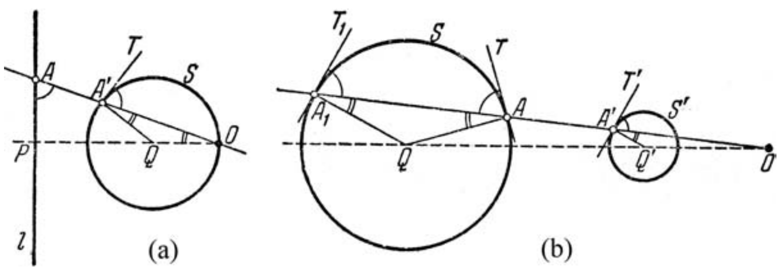


FIGURE 13

Assume that one of the two figures is a line passing through the center of inversion. If the second figure is also a line, then for proof consider Figure 13a, where Q is the center of the circle S which is the image of the (second) line l and $A'T$ is the tangent to S at A'. It is clear that

$$\angle TA'O = 90° - \angle OA'Q = 90° - \angle A'OQ = \angle PAO$$

(see the proofs of properties B_2 and B_3 of inversion). If the second figure is a circle, then for proof consider Figure 13b, where Q and Q' are the respective centers of the circles S and S', AT is the tangent to S at A, and $A'T'$ is the tangent to S' at A'. It is clear that

$$\angle Q'A'O = \angle QA_1O = \angle QAA_1,$$

$$\angle TAA_1 = 90° - \angle QAA_1 = 90° - \angle QA_1A$$
$$= 90° - \angle Q'A'O = \angle T'A'O$$

(see the proof of property B_4 of inversion).

Now consider the case of two circles S_1 and S_2 which intersect in a point A. Let their images under inversion be circles S_1' and S_2' which intersect in a point A'. Let AT_1 and AT_2 be the tangents to S_1 and S_2 at A, and let $A'T_1'$ and $A'T_2'$ be the tangents to S_1' and S_2' at A' (Figure 14). From what was

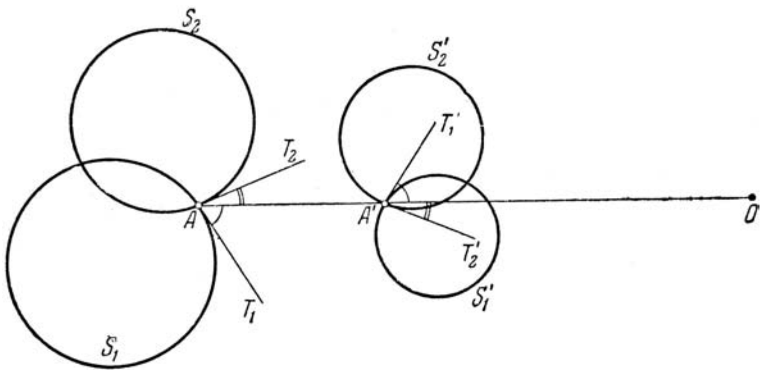


FIGURE 14

proved earlier it follows that

$$\angle T_1'A'O = \angle T_1AO \quad \text{and} \quad \angle T_2'A'O = \angle T_2AO.$$

But then

$$\angle T_1'A'T_2' = \angle T_1AT_2,$$

which is what we wished to prove.

The angle between two tangent circles (or a line and a circle) is equal to zero. If the point of tangency is not the center of inversion, then property C implies that *the images under inversion of two tangent circles and of a line tangent to a circle are, respectively, two tangent circles and a line tangent to a circle.* If the point of tangency is the center of inversion, then each of the images in question is a pair of parallel lines (Figure 15).

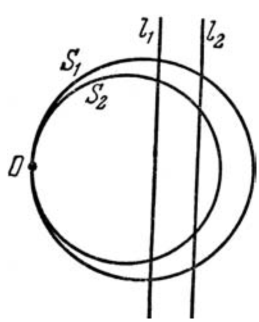


FIGURE 15

Inversion *reverses the direction of the angle between two circles* (or between two lines, or between a line and a circle). (In this it resembles a reflection in a line.) Specifically, let S_1 and S_2 be two circles which intersect in a point A. Let the respective tangents at A to these circles be AT_1 and AT_2 and let AT_1 go over to AT_2 under a counterclockwise rotation through α about A. Then the image of this configuration under an inversion consists of circles S_1' and S_2' which intersect in a point A', and the tangent $A'T_1'$ to S_1' at A' goes over to the tangent $A'T_2'$ to S_2' at A' under a clockwise rotation through an angle α about A' (Figures 16a and 16b).

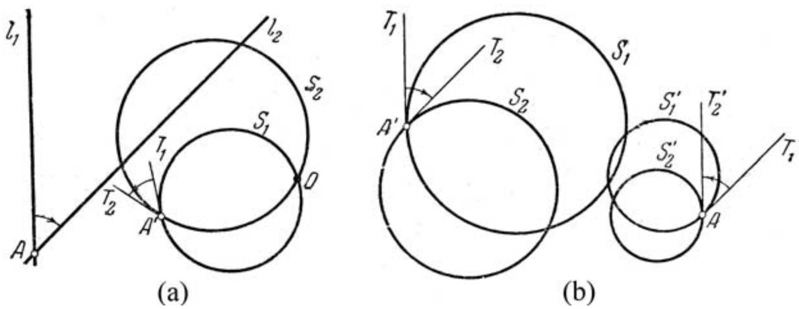


FIGURE 16

By a tangent to a curve γ at a point M we mean the limiting position of a secant MM_1 when M_1 tends to M (Figure 17a). By the angle between two curves which intersect in a certain point we mean the angle between their tangents at that point (Figure 17b). It is easy to show that *inversion takes two intersecting curves to two new intersecting curves which form the same angle as the original curves* (in other words, *inversion preserves angles between curves*).[7]

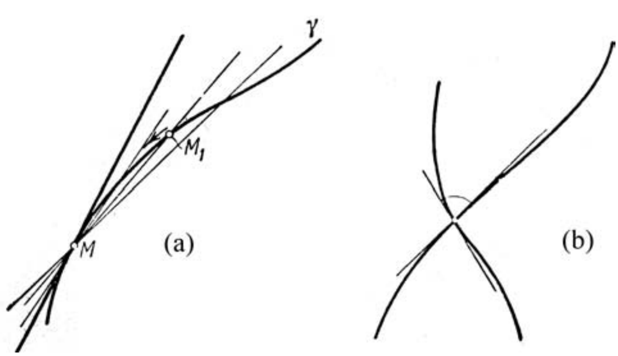


FIGURE 17

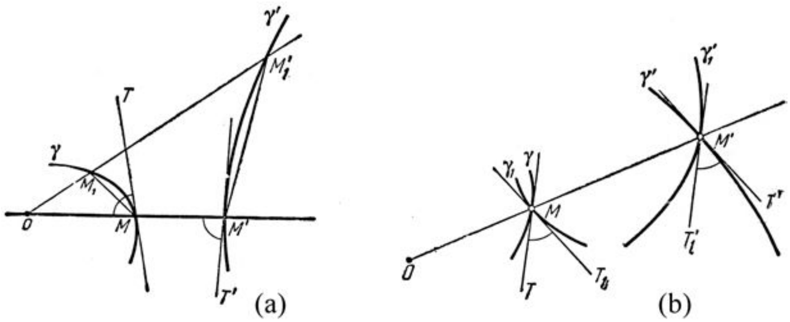


FIGURE 18

In fact, let γ be a curve and let γ' be its image under an inversion with center O and power k. Let M and M_1 be two close points on γ with respective images M' and M_1' on γ' (Figure 18a). Since $OM/OM_1 = OM_1'/OM'$ (this equality follows from $OM \cdot OM' = OM_1 \cdot OM_1' = k$), the triangles OMM_1 and $OM_1'M'$ are similar. Hence $\angle M_1MO = \angle M'M_1'O$. If M_1 tends to M, then the angle OMM_1 tends to the angle OMT formed by the tangent MT to γ at M and the line OM, and the angle $OM_1'M'$ tends to the angle $OM'T'$ formed by the tangent $M'T'$ to γ' at M' and the line OM'. Arguing as we did in proving property C, we deduce from the equality of the angles formed by the tangents MT and $M'T'$ to γ and γ' with the line OMM' (Figure 18a) that inversion preserves the angle between curves (Figure 18b).

We note that properties B and C of inversion (reflection in a circle) hold for reflection in a line. Property A of a reflection in a circle is analogous to the following property of a reflection in a line: *A reflection in a line l fixes l and interchanges the halfplanes into which l divides the plane.*

The solutions of the problems below are based on properties A, B, and C of inversion. For refinements required in particular problems see p. 28f.

1. Let S be a circle tangent to two circles S_1 and S_2. Show that the line joining the points of tangency passes through a center of similarity of the two circles.

This problem appears, in another context, as Problem 21 in NML 21.

2. Let A and B be two given points on a circle S. Consider all pairs of circles S_1 and S_2 tangent to S at A and B respectively and

(a) tangent to each other;

(b) perpendicular to each other.

In (a) find the locus of the points of tangency and in (b) the locus of the points of intersection of S_1 and S_2.

3. Let A, B, C, D be four points that are neither collinear nor concyclic. Show that the angle between the circles circumscribed about the triangles ABC and ABD is equal to the angle between the circles circumscribed about the triangles CDA and CDB.

4. Let S_1, S_2, and S be semicircles with collinear diameters AM, MB, and AB (Figure 19). At M draw the perpendicular MD to the line AB and inscribe circles Σ_1 and Σ_2 in the curvilinear triangles ADM and BDM. Show that

(a) Σ_1 and Σ_2 are congruent;

(b) the common tangent to Σ_1 and S_1 at their point of tangency passes through B and the common tangent to Σ_2 and S_2 at their point of tangency passes through A.

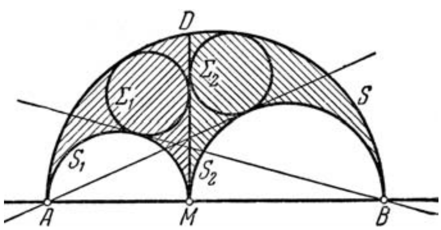


FIGURE 19

5. Show that if each of four circles S_1, S_2, S_3, and S_4 is tangent to two of its neighbors (for example, the neighbors of S_1 are S_2 and S_4), then the four points of tangency are concyclic; in Figure 20 they lie on the circle Σ.

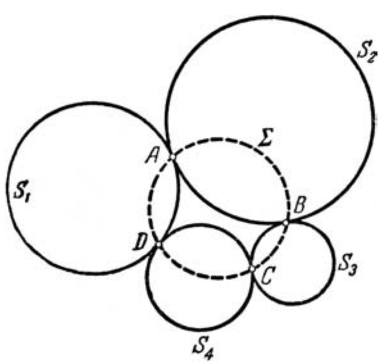
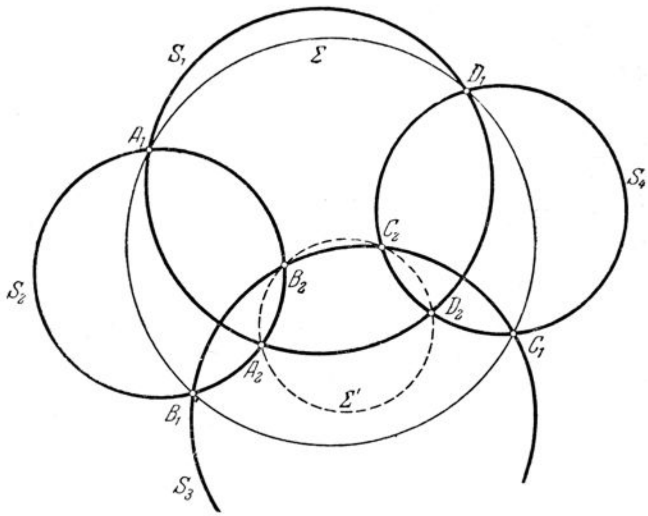


FIGURE 20

6. (a) Given six coplanar points A_1, A_2, A_3 and B_1, B_2, B_3. Show that if the circles circumscribed about the triangles $A_1A_2B_3$, $A_1A_3B_2$, and $A_2A_3B_1$ intersect in a single point, then so do the circles circumscribed about the triangles $B_1B_2A_3$, $B_1B_3A_2$, and $B_2B_3A_1$ (see Figure 38a on p. 30)

(b) Given four circles S_1, S_2, S_3, and S_4 that intersect in pairs: S_1 and S_2 in points A_1 and A_2, S_2 and S_3 in points B_1 and B_2, S_3 and S_4 in points C_1 and C_2, and S_4 and S_1 in points D_1 and D_2 (Figure 21a). Show that if A_1, B_1, C_1, and D_1 lie on one circle (or line) Σ, then A_2, B_2, C_2, and D_2 also lie on one circle (or line) Σ'.



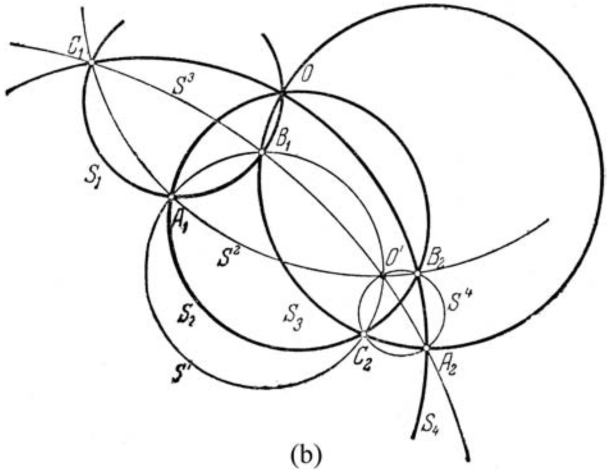
(a)

FIGURE 21

(c) Given six coplanar points A_1, A_2; B_1, B_2; C_1, C_2. Show that if the circles S_1, S_2, S_3, and S_4 circumscribed about the triangles $A_1B_1C_1$, $A_1B_2C_2$, $A_2B_1C_2$, and $A_2B_2C_1$ intersect in one point (O in Figure 21b), then so do the circles S^1, S^2, S^3, and S^4 circumscribed about the triangles $A_1B_1C_2$, $A_1B_2C_1$, $A_2B_1C_1$, and $A_2B_2C_2$ (O' in Figure 21b).

7. (a) We will say of n lines in a plane that they are in general position if no two of them are parallel and no three of them are concurrent.

We will call the point of intersection of two lines in general position (that is, two intersecting lines) their central point (Figure 22a).



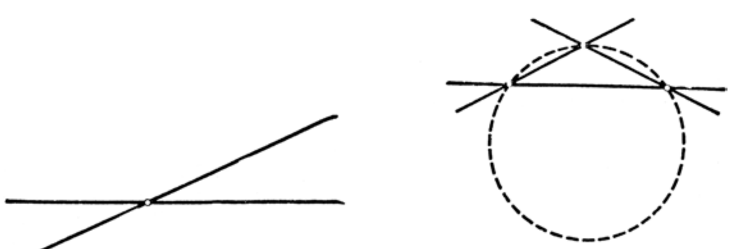
(b)

FIGURE 21

By successively removing each of three lines in general position we obtain three pairs of lines. We call the circle passing through the three central points of these three pairs of lines the central circle of the three lines. [Obviously, the central circle of three lines is simply the circle circumscribed about the triangle formed by these lines (Figure 22b).]

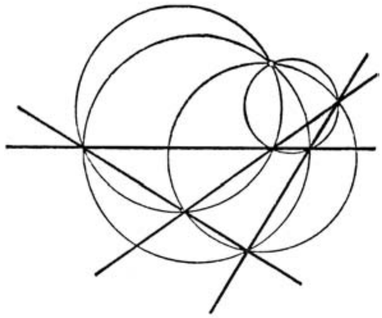
By successively removing each of four lines in general position we obtain four triples of lines. Show that the four central circles of these triples of lines intersect in a single point (Figure 22c).[8] We call this point the central point of four lines.

By successively removing each of five lines in general position we obtain five quadruples of lines. Show that the five central points of these quadruples

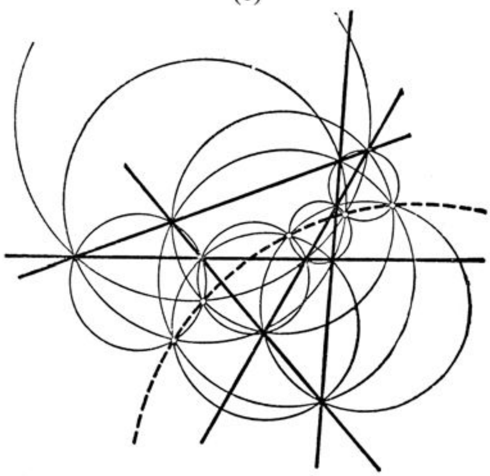


(a) (b)

FIGURE 22



(c)



(d)

FIGURE 22

of lines lie on a single circle (Figure 22d). We call this circle the central circle of five lines.[9]

Quite generally, by successively removing each of n lines in general position, n odd, we obtain n central points of the groups of $n - 1$ lines. Show that these n central points lie on a single circle, the central circle of n lines. To each n lines in general position, n even, there correspond n central circles associated with all possible groups of $n - 1$ lines. Show that these n circles always intersect in a single point, the central point of n lines.

(b) The circle passing through two points on two intersecting lines and through their point of intersection is called the directing circle of the two lines (Figure 24a).

Now consider three lines in general position with a point given on each line. By successively removing each of the three lines we obtain three pairs of lines. Show that the directing circles of these three pairs of lines intersect in a single point, the directing point of three lines (Figure 24b).[10]

Next we consider four lines in general position with a point given on each line. We require these points to be concyclic. By successively removing each of the four lines we obtain four triples of lines, and thus four directing points of the four triples. Show that these four points lie on a circle, the directing circle of four lines (Figure 24c).[11]

Quite generally, to an odd number n of lines in general position on each of which we are given a point such that the given points are concyclic there correspond n circles, the directing circles of the groups of $n-1$ lines with a given point on each line, obtained by successively removing one of the

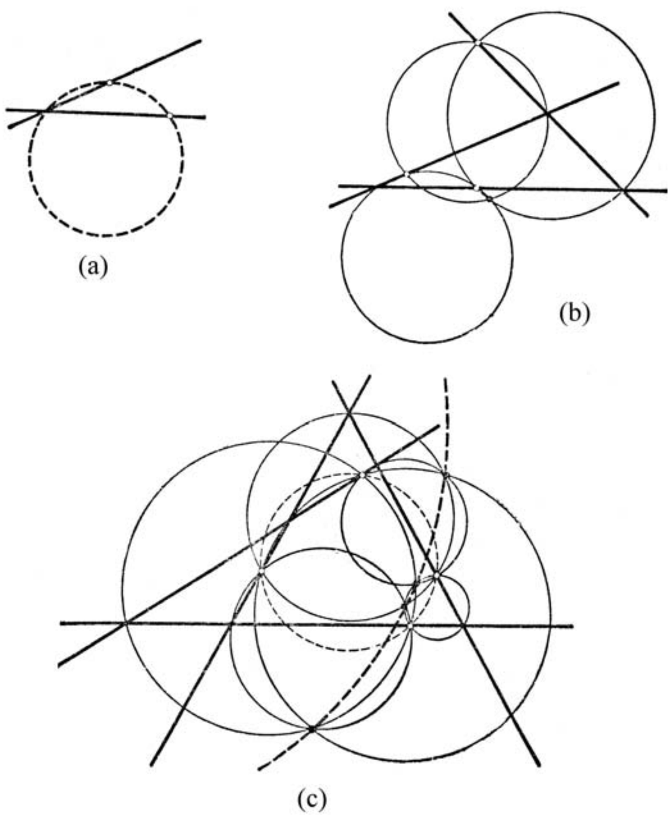


FIGURE 24

given n lines. Show that these n circles intersect in a single point, called the directing point of n lines. To an even number n of lines in general position on each of which we are given a point such that the given points are concyclic there correspond n points, the directing points of the groups of $n-1$ of our lines. Show that these n points lie on a circle, the directing circle of n lines.

In problems 7(a) and 7(b) it is assumed that the given n lines are in general position. We note that this assumption is not essential. In this connection see pp. 28–29 below.

8. (a) Prove the following relation connecting the radii R and r of the circumscribed and inscribed circles of a triangle and the distance d between their centers:

$$\frac{1}{R+d}+\frac{1}{R-d}=\frac{1}{r}. \tag{$*$}$$

Conversely, if the radii R and r of two circles and the distance d between their centers satisfy the relation $(*)$, then these circles can be regarded as the circumscribed and inscribed circles of a triangle (in fact, of an infinite number of triangles; as a vertex of such a triangle we can take any point of the larger circle (see Figure 26)).

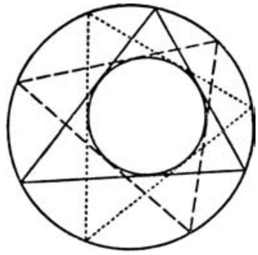


FIGURE 26

(b) Prove the following relation connecting the radii R and r_1 of the circumscribed and escribed circles of a triangle and the distance d_1 between their centers:

$$\frac{1}{d_1-R}-\frac{1}{d_1+R}=\frac{1}{r_1}.$$

Remark. The formulas in Problems 8(a) and 8(b) can be rewritten in the following simple form

$$d^2=R^2-2Rr \quad \text{and} \quad d_1^2=R^2+2Rr_1.$$

9. (a) A quadrilateral $ABCD$ is inscribed in a circle and circumscribed about another circle. Show that the lines joining the points of tangency of the opposite sides of the quadrilateral with the inscribed circle are perpendicular.

(b) Show that the radii R and r of the circles circumscribed about and inscribed in the quadrilateral $ABCD$ and the distance d between their centers are connected by the relation

$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}. \qquad (**)$$

Conversely, if the radii R and r of two circles and the distance d between their centers satisfy the relation $(**)$, then these circles can be regarded as the circumscribed and inscribed circles of a quadrilateral (in fact, of an infinite number of quadrilaterals; as a vertex of such a quadrilateral we can take any point of the larger circle (see Figure 27)).

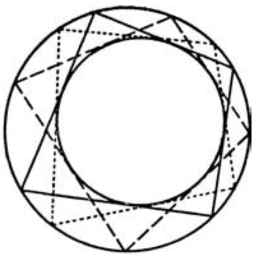


FIGURE 27

The theorems in problems 8(a) and 9(b) show that either there is no n-gon ($n = 3$ or $n = 4$) inscribed in one of two given circles and circumscribed about the other or there are infinitely many such n-gons. One can show that this proposition holds also for any $n > 4$ (see problem 41 in Section 3, p. 59).

10. Show that the radius r of the inscribed circle of a triangle cannot exceed half the radius R of the circumscribed circle, and that the equality $r = R/2$ holds if and only if the triangle is equilateral.

11. Show that the nine point circle of a triangle (see problem 17(a) in NML 21) is tangent to the inscribed circle as well as to the three escribed circles (Figure 28).

See also Problem 63 in Section 5.

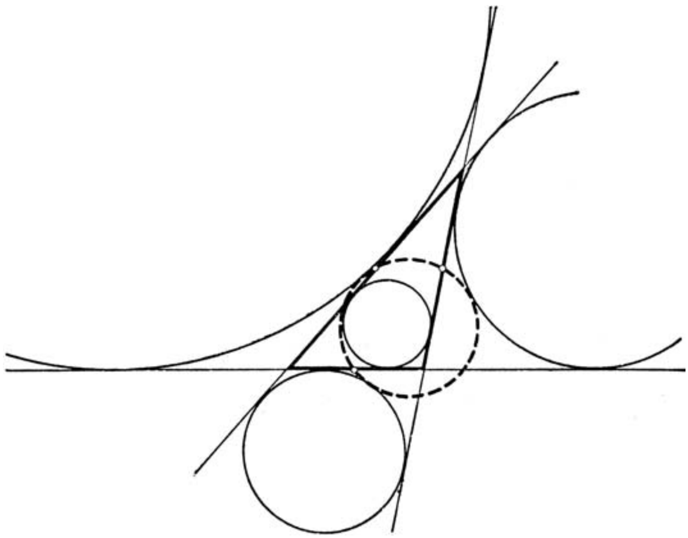


FIGURE 28

12. Use the properties of inversion to deduce Pascal's theorem (problem 46, p. 62 in NML 24) from the theorem on three centers of similarity (p. 29, NML 21).

We will now prove a theorem which will be frequently used when solving subsequent problems in this book.

Theorem 2. *Any two circles, or a line and a circle, can be transformed by an inversion into two* (*intersecting or parallel*) *lines or two concentric circles.*

Proof. A circle S can always be inverted into a line by using any of its points as a center of inversion (see p. 11). If we invert a line using any of its points as the center of inversion, then the line goes over to itself. It follows that if two circles, or a line and a circle, have a point in common, then an inversion with the common point as center takes either pair of figures to a pair of lines. More specifically, if the figures of a pair are tangent to one another, then the two image lines are parallel (Figure 29b), and if they share another point, then the two image lines intersect in its image under the inversion (Figure 29a).

It remains to show that *two nonintersecting circles, or a line and a circle without common points, can be inverted into two concentric circles.* The proof follows.

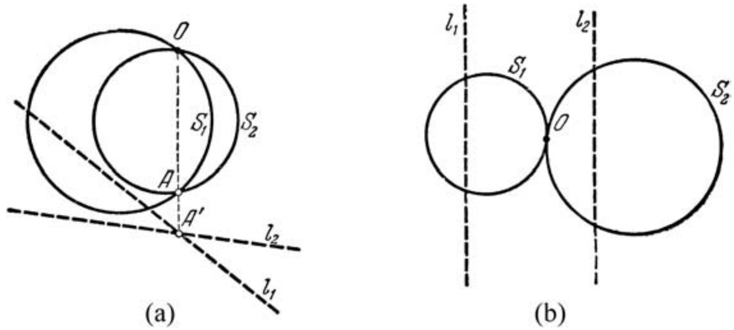


FIGURE 29

Suppose that a circle S and a line l have no points in common (Figure 30). Drop a perpendicular o from the center of S to l and denote its foot by P. Draw a circle $\overline{S}$ with center P and radius whose length is that of the tangent PQ from P to S; it is clear that $\overline{S}$ is perpendicular to both S and l. Consider an inversion whose center O is one of the points of intersection of $\overline{S}$ and o. Under this inversion o goes over to itself (property B_1 of inversion) and $\overline{S}$ goes over to a line $\overline{S}'$ (property B_3). The line l and the circle S go over to two circles l' and S' (properties B_2 and B_4) both of which are perpendicular to o as well as to $\overline{S}'$ (property C). But if a circle is perpendicular to a line, then its center lies on that line (see p. 3). Hence the centers of the circles l' and S' are on the lines o and $\overline{S}'$, that is, coincide with their point of intersection. This shows that, as asserted, our inversion takes the circle S and the line l to two concentric circles.

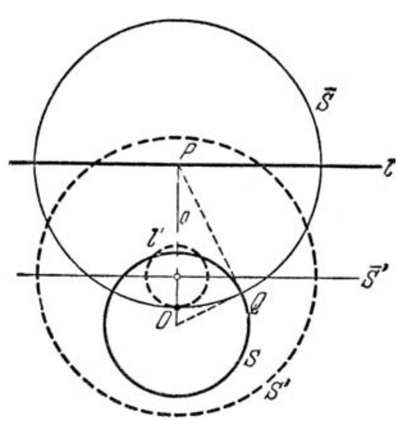


FIGURE 30

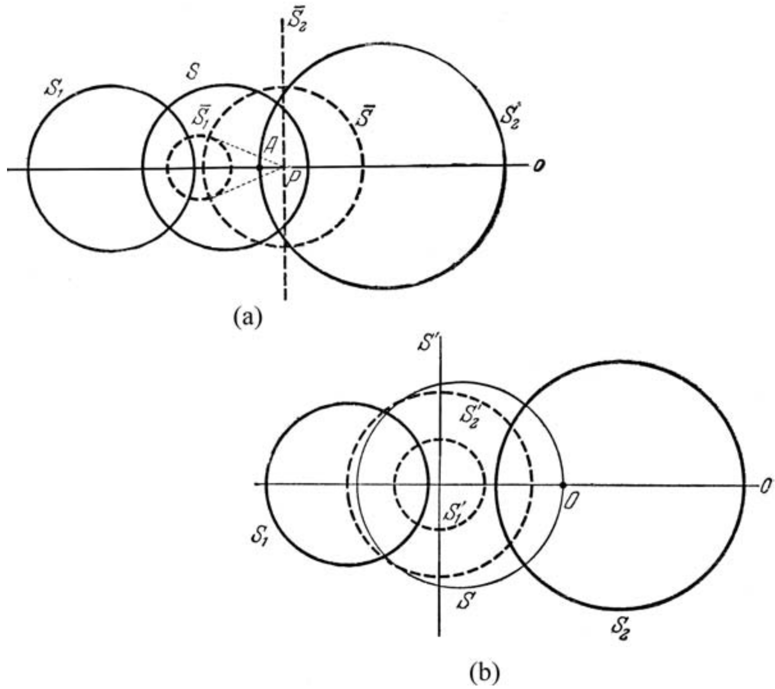


FIGURE 31

One can show by a similar argument that two nonintersecting circles S_1 and S_2 can be taken by inversion to two concentric circles. First we show that there is a circle S with center on the line o determined by the centers of our circles that is perpendicular to both S_1 and S_2. To this end we use an inversion whose center A is a point of intersection of S_2 and o. This inversion takes the disjoint circles S_1 and S_2 to two disjoint figures, namely a circle $\overline{S}_1$ and a line $\overline{S}_2$ (Figure 31a). Let $\overline{S}$ be the circle whose center is the point of intersection of the lines o and $\overline{S}_2$ and the length of whose radius is equal to the length of the tangent from P to $\overline{S}_1$; the circle S is the preimage of $\overline{S}$ under our inversion. Since $\overline{S}$ is perpendicular to both $\overline{S}_1$ and $\overline{S}_2$ (see above), it follows (by property C of inversion) that S is perpendicular to both S_1 and S_2.

All we need do now is apply an inversion whose center O is a point of intersection of S and o. One shows, just as was done before, that this inversion takes the circles S_1 and S_2 to two concentric circles S_1' and S_2' (whose common center is the point of intersection of the line o and the line S', which is the image of the circle S under the inversion; see Figure 31b).

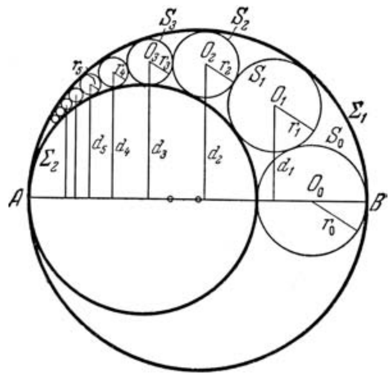


FIGURE 32

13. Let Σ_1 and Σ_2 be two internally tangent circles. A sequence of circles $S_0, S_1, S_2, \ldots$ is inscribed in the figure formed by Σ_1 and Σ_2 (Figure 32). The center of S_0 lies on the line AB determined by the centers of Σ_1 and Σ_2 and S_n is tangent to S_{n-1} $(n = 1, 2, 3, \ldots)$. Let $r_0, r_1, r_2, r_3, \ldots$ denote the radii of the circles $S_0, S_1, S_2, S_3, \ldots$ and $d_0, d_1, d_2, d_3, \ldots$ the distances of their centers from the line AB.

(a) Show that $d_n = 2nr_n$.

(b) Express the radius r_n of S_n in terms of the radii R_1 and R_2 of Σ_1 and Σ_2 and the number n.

14. Let Σ_1 and Σ_2 be two intersecting circles. Let $S_1, S_2, S_3, \ldots$ be circles inscribed in the lune formed by Σ_1 and Σ_2 (Figure 33). Let $r_1, r_2, r_3, \ldots$ denote the radii of the circles $S_1, S_2, S_3, \ldots$ and $d_1, d_2, d_3, \ldots$

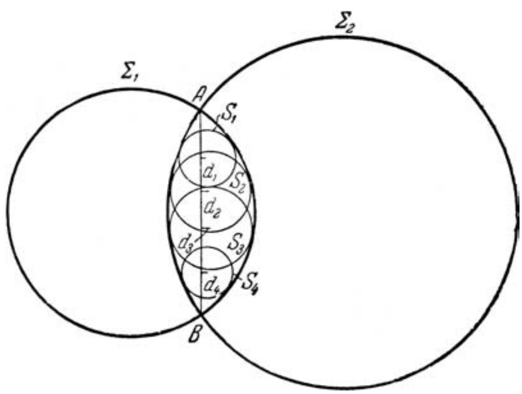


FIGURE 33

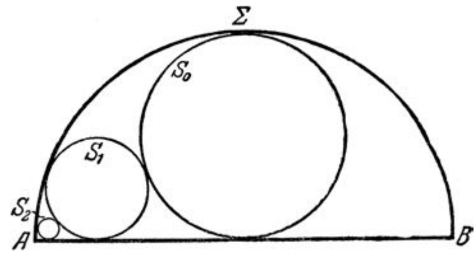


FIGURE 34

the distances of their centers from the common chord AB of Σ_1 and Σ_2. Show that

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \frac{r_3}{d_3} = \cdots = \sigma.$$

What is the geometric significance of σ?

15. Let $S_0, S_1, S_2, S_3, \ldots$ be circles internally tangent to the semicircle Σ and to its diameter AB (Figure 34). Denote by $R, r_0, r_1, r_2, r_3, \ldots$ the radii of the circles $\Sigma, S_0, S_1, S_2, S_3, \ldots$ and by $1/t_0, 1/t_1, 1/t_2, \ldots$ the ratios $r_0/R,\ r_1/R,\ r_2/R, \ldots.$

(a) Express the radius r_n of S_n in terms of the radius R and the number n.

(b) Show that

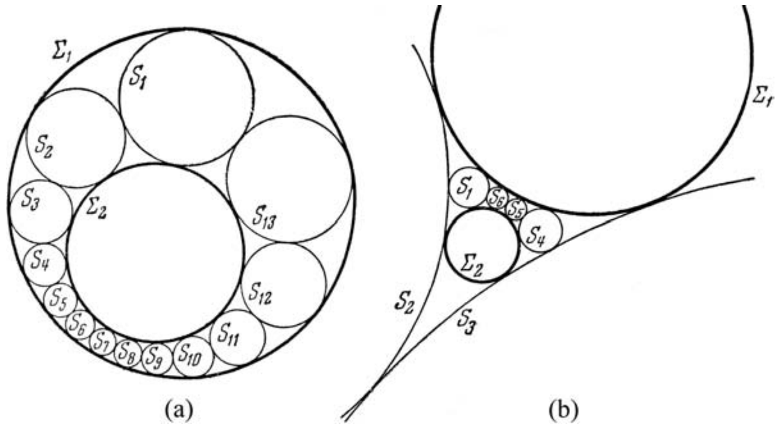
$$t_0 = 2, t_1 = 4, t_2 = 18, t_3 = 100, t_4 = 578, \ldots$$

and that, more generally, $t_n (n \geq 2)$ is an integer expressible in terms of t_{n-1} and t_{n-2} by the simple formula

$$t_n = 6t_{n-1} - t_{n-2} - 4.$$

16. A *chain* is a set of finitely many circles $S_1, S_2, \ldots, S_n$ each of which is tangent to two nonintersecting circles Σ_1 and Σ_2 (called the *base* of the chain) and to two other circles (see Figures 35a and 35b). Clearly, if one of the circles Σ_1 and Σ_2 is in the interior of the other, then the circles of the chain are tangent to them in different ways (externally to one and internally to the other; Figure 35a); and if each of the circles Σ_1 and Σ_2 is in the exterior of the other, then the circles of the chain are tangent to them in the same way (externally or internally to both; Figure 35b). Prove that

(a) If Σ_1 and Σ_2 are the base of a chain, then they are the base of infinitely many chains each of which contains as many circles as the initial chain. [More precisely, any circle tangent to Σ_1 and Σ_2 (in different ways

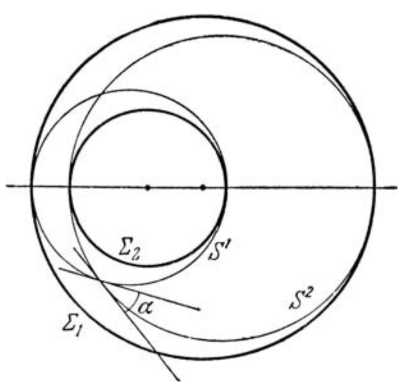


(a) (b)

FIGURE 35

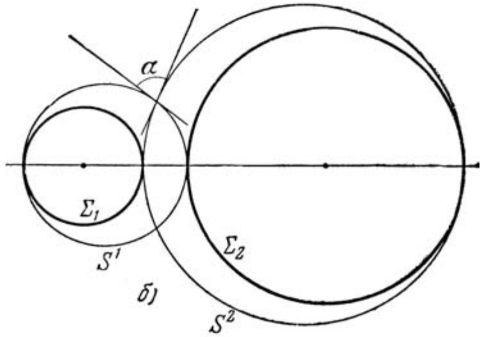
if one of these circles is in the interior of the other, and in the same way if each of them is in the exterior of the other) can be included in a chain with base Σ_1, Σ_2.]

(b) For two nonintersecting circles Σ_1 and Σ_2 to be the base of a chain (and thus of infinitely many chains; see problem (a)), it is necessary and sufficient that the angle α, between circles S^1 and S^2 tangent to Σ_1 and Σ_2 at the points of their intersection with the line determined by their centers, be commensurable with 360°; here S^1 and S^2 must be tangent to Σ_1 and Σ_2 in the same way if one of the latter circles is in the interior of the other (Figure 36a) and in different ways if each of them is in the exterior of the



(a)

FIGURE 36



(b)

FIGURE 36

other (Figure 36b). More specifically, if

$$\alpha = \frac{m}{n} \cdot 360^\circ,$$

then Σ_1 and Σ_2 can serve as the base of a chain of n circles. Moreover, the points of tangency of the circles of the chain with Σ_1 (or with Σ_2), taken in the same order as the order of the circles in the chain, traverse Σ_1 (or Σ_2) m times (in Figures 35a and 35b, $m = 1$).

The fraction m/n is called the characteristic of the chain.

(c) Suppose that a chain with base Σ_1, Σ_2 contains an even number of circles $S_1, S_2, \ldots, S_{2n}$. Then S_1 and S_{n+1}, the "antipodal" circles of the chain, can also serve as the base of a chain (Figure 37). Moreover, if the characteristic of the chain with base Σ_1, Σ_2 is m/n (see part (b)) and the characteristic of the chain with base S_1, S_{n+1} is m'/n', then

$$\frac{m}{n} + \frac{m'}{n'} = \frac{1}{2}.$$

We conclude this section with the observation that the existence in the plane of a special ("distinguished") point (center of inversion), which does not go over to any point under inversion, calls for certain—hitherto ignored—stipulations. For example, the condition stated in problem 6(a) is not quite correct: the fact that the circles circumscribed about the triangles $A_1A_2B_3$, $A_1A_3B_2$, and $A_2A_3B_1$ intersect in a single point does not imply the existence of circles circumscribed about the triangles $B_1B_2A_3$, $B_1B_3A_2$, and $B_2B_3A_1$—it is possible that one of the point-triples B_1, B_2, A_3; B_1, B_3, A_2; and B_2, B_3, A_1 (or possibly all three of them) is collinear. The

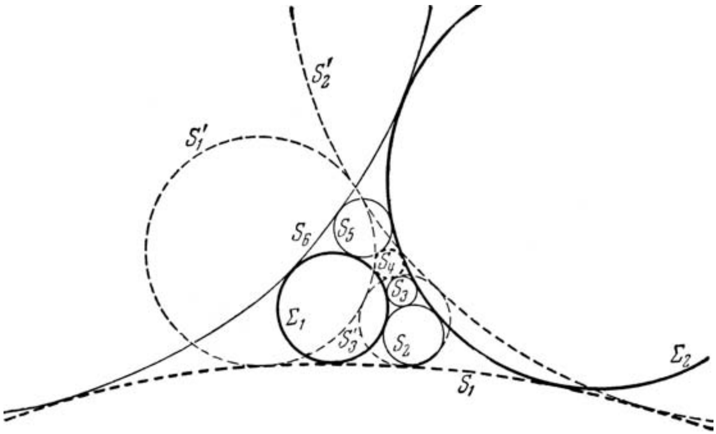


FIGURE 37

following is a correct formulation of the problem: *If the circles circumscribed about the triangles* $A_1A_2B_3$*,* $A_1A_3B_2$*, and* $A_2A_3B_1$ *intersect in a single point, then either the circles circumscribed about the triangles* $B_1B_2A_3$*,* $B_1B_3A_2$*, and* $B_2B_3A_1$ *intersect in a single point* (Figure 38a)—*here one or two of these circles can be circles of* "*infinite radius*," *that is, lines* (Figure 38b)—*or each of the point-triples* B_1, B_2, A_3*;* B_1, B_3, A_2*; and* B_2, B_3, A_1 *is collinear* (Figure 38c). Similar stipulations are required in the statements of some of the earlier problems.

We can dispense with the need for such stipulations by proceeding as in the case of projective transformations (see NML 24, p. 37 f.), namely by introducing a fictitious "point at infinity" which is mapped under an inversion in a circle Σ on its center O and which is the image of O under this inversion.[12] [Note that by making appropriate conventions we can extend property D of inversion, stated in the sequel (see Section 4, p. 63), to the case when one of the points, whether initial or transformed, is the "point at infinity"—see NML 24, pp. 39 and 40.] One should also keep in mind that in discussing inversions the term "circle" stands for what we usually mean by a circle as well as for a line (which can be thought of as a "circle of infinite radius" or as a "circle passing through the point at infinity").

Speaking of problem 6(a). As was just pointed out, we can assume that one, two, or all three of the point-triples A_1, A_2, B_3; A_1, A_3, B_2; and A_2, A_3, B_1 are collinear; in the latter case we need not require that the three lines intersect in a point (any three lines intersect in the "point at infinity").

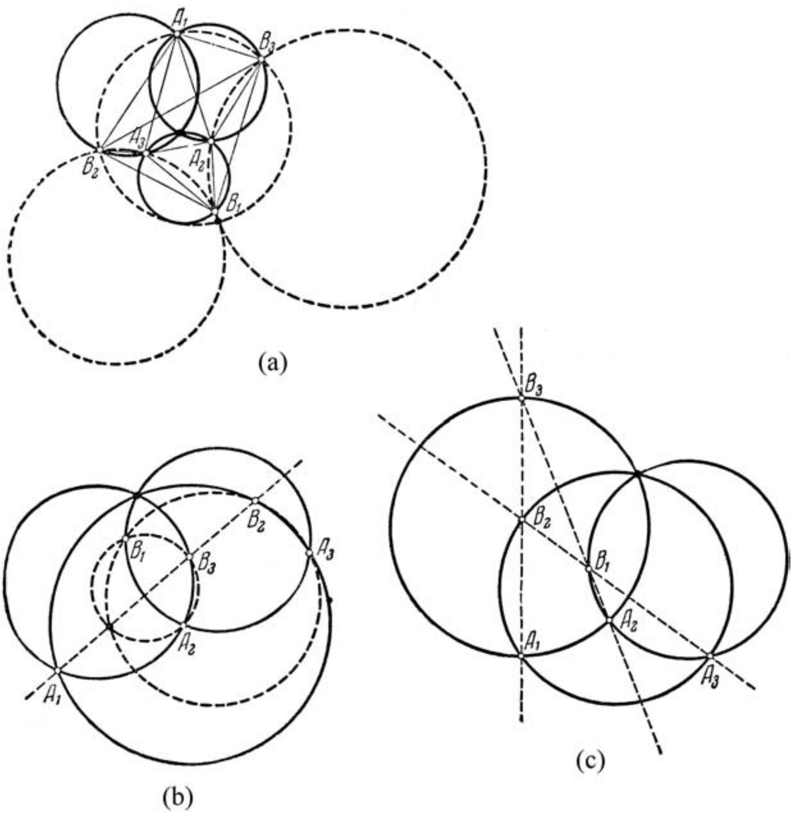


FIGURE 38

Introduction of these conventions also does away with the need to make stipulations concerning the disposition of lines in problems 7(a) and 7(b); in addition, in problem 7, part (b) we can assume that the points chosen on the lines are either concyclic or collinear. Specifically, if the three lines l_1, l_2, and l_3 intersect in a point A, then this point must be viewed as their central circle (a "circle of zero radius"); if two of the three lines l_1, l_2, and l_3 are parallel, then the third of these three lines must be viewed as their central circle (a "circle of infinite radius"); and if all three lines l_1, l_2, and l_3 are parallel, then the point at infinity of the plane must be viewed as their central circle. If these conventions are adopted, then the theorem in problem 7(a) holds regardless of whether the n lines under consideration are in general position or not; the central point of $n = 2k$ lines can be finite or at infinity; and the central circle of $n = 2k + 1$ lines can be an ordinary circle, a point (a "circle of zero radius"; in particular, it can be the "point at infinity" of the plane), or a line (a "circle of infinite radius").

Notes to Section 1

[1] If the circles S_1 and S_2 intersect in two points A and B then, obviously, the angle between the tangents to S_1 and S_2 at A is equal to the angle between the tangents to S_1 and S_2 at B (Figure 3a). Similarly, if a line l and a circle S intersect in two points A and B, then the tangents to S at A and B form equal angles with l (Figure 3b).

[2] It is easy to see that if two points A and A' are symmetric with respect to a circle S, then any circle passing through A and A' is perpendicular to S (this is an analogue of the fact that the center of a circle passing through two points A and A' symmetric with respect to a line l is on that line).

[3] Basically, this new definition of a reflection in a circle is quite close to the definition of a reflection in a line adopted in Section 1, Ch. II of NML 8.

[4] This definition explains why inversion is sometimes referred to as a transformation by reciprocal radii. It also justifies the very term "inversion" (from the Latin "inversio," transposition).

[5] Compare this with the definition of a central similarity with negative similarity coefficient in Section 1, Ch. I of NML 21.

[6] It is easy to see that if O is in the interior of S, or if the degree of the inversion is negative (these two things cannot occur simultaneously), then the similarity coefficient of S and S' is negative (it is equal to $-\frac{k}{k'}$), that is, A' and A_1 are on different sides of O.

[7] A transformation that preserves angles between curves is said to be conformal. Hence *inversion is a conformal transformation.*

[8] See Problem 35 in NML 21.

[9] It is not difficult to see that this proposition is analogous to the following: *Extend the two neighboring sides of each side of a* (not necessarily convex) *pentagon until they intersect and circumscribe a circle about each of the five resulting triangles. Then the five points of intersection of neighboring circles lie on a circle*, the so-called central circle of the sides of the pentagon (Figure 23).

[10] See Problem 58(a) in NML 21.

[11] It is not difficult to see that this proposition is equivalent to the following: *If we take on the sides of an arbitrary quadrilateral four concyclic points, join each of them to its immediate neighbors, and circumscribe circles about each of the four resulting triangles, then the four points of*

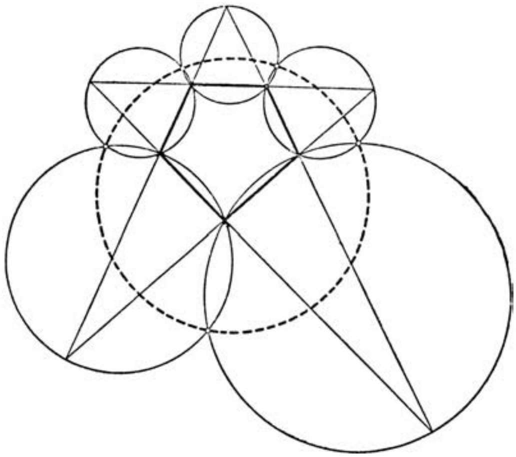


FIGURE 23

intersection of neighboring circles lie on a circle, the directing circle of the sides of the quadrilateral (Figure 25).

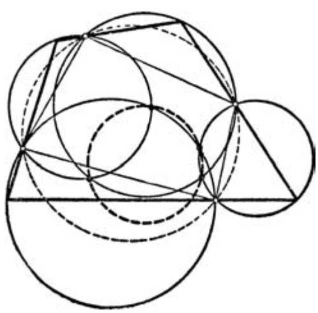


FIGURE 25

[12] The term “point at infinity” is connected with the fact that if a point M comes arbitrarily close to the center O of a circle Σ, then the point M', symmetric to M with respect to Σ, moves indefinitely far from O.

A plane supplemented by the addition of a fictitious “point at infinity” is called a conformal, or analagmatic, plane (compare this with the definition of the projective plane on pp. 41 and 42 in NML 24).

2

Application of inversions to the solution of constructions

Problems. Constructions with compass alone

In this section we consider a number of construction problems whose solution is simplified by the use of inversions. We will rely on the fact that a figure obtained from a given figure by an inversion with given center O and power k can always be constructed with ruler and compass. For example, Figure 39 shows the construction of the point A' symmetric to a given point A with respect to a given circle Σ with center O. In fact, the similarity of the triangles OAP and OPA' implies that

$$OA/OP = OP/OA',$$

and thus

$$OA \cdot OA' = OP^2.$$

More specifically, to construct A' when A is outside Σ we draw tangents AP and AQ to Σ from A, and A' is the point of intersection of the lines OA and PQ. To construct A' when A is inside Σ we draw through A the

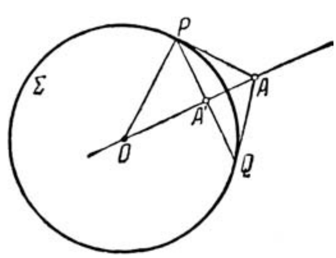


FIGURE 39

perpendicular to OA and then the tangents to Σ at the points P and Q in which that perpendicular meets Σ. The required point A' is the point of intersection of OA and the tangents to Σ at P and Q (think of having interchanged the positions of A and A' in Figure 39).

To construct a circle or line S' symmetric to a given circle or line S with respect to a given circle Σ with center O it suffices to find three points M', N', and P' symmetric to any three points M, N, and P of S. An even simpler approach is to make use of the proofs of properties B_1–B_4 of inversion (see pp. 8–9). If S is a line passing through O, then S' coincides with S. If S is a line not passing through O, then a diameter of the circle S' is the segment OP', where P' is symmetric with respect to Σ to P, the foot of the perpendicular from O to S (see Figure 10a). If S is a circle passing through O and OP is a diameter of S, then S' is the line perpendicular to OP and passing through the point P' symmetric to P with respect to Σ (see Figure 10b). Finally, if S is a circle not passing through O and A is one of its points, then S' is the circle which is centrally similar to S with respect to the center of similarity O and passes through the point A' symmetric to A with respect to Σ (see Figure 11).[1]

We note that using ruler and compass it is easy to realize an inversion that takes a given circle to a line; or two given circles (or a line and a circle) to two lines or to two concentric circles (see Theorem 2, Section 1). In fact, to take a given circle S to a line choose a point of S as the center of inversion. To take two circles (or a line and a circle) S_1 and S_2 to two lines choose a common point of these circles as the center of inversion. To take a circle S and a line l disjoint from S to two concentric circles choose as the center of inversion the point of intersection of the perpendicular OP, dropped from the center of S to l, with the circle $\overline{S}$ with center at P and radius equal to the tangent from P to S (see Figure 30). To take nonintersecting circles S_1 and S_2 to two concentric circles choose as the center of inversion a point of intersection of the line of centers O_1O_2 of the circles S_1 and S_2 with any circle S perpendicular to S_1 and S_2. Such an inversion takes S and O_1O_2 to two lines and the circles S_1 and S_2 to circles S_1' and S_2' perpendicular to these two lines, that is, circles with common center coincident with the point of intersection of the lines (see the proof of Theorem 2 in Section 1). To construct a circle S perpendicular to the circles S_1 and S_2 we construct first a circle $\overline{S}$ perpendicular to the circle $\overline{S_1}$ and the line $\overline{S_2}$, the images of the circles S_1 and S_2 under an inversion whose center A is the point of intersection of O_1O_2 and S_2; the center P of $\overline{S}$ is the point of intersection of O_1O_2 and $\overline{S_2}$ and its radius is equal to the tangent from P to S_2 (see

Figure 31a); the required circle S is the image of the circle $\overline{S}$ under the inversion that takes S_1 and S_2 to $\overline{S_1}$ and $\overline{S_2}$ (see Figure 31b). [2]

An inversion with negative power is also easy to realize with ruler and compass. This is so because such an inversion is equivalent to an inversion with positive power (a reflection in a circle) followed by a reflection in a point (see p. 6).

17. Given an angle MAN and a point O not on its sides. Draw a line through O that intersects the sides of the angle in points X and Y such that the product $OX \cdot OY$ has a given value k.

18. Inscribe in a given parallelogram a parallelogram with given area and given angle between its diagonals.

19. Given three points A, B, and C draw a line l through A such that:

(a) the product of the distances from B and C to l has a given value;

(b) the difference of the squares of the distances from B and C to l has a given value.

20. Let S be a given circle. Inscribe in S an n-gon whose sides pass through n given points (or: some of the sides of the n-gon have given directions and the remaining sides pass through given points). Consider separately the case of an even n and an odd n.

Problem 20 appears in another connection in NML 24 (see problem 84, p. 97).

Problems involving the construction of circles

21. Draw a circle

(a) passing through two given points A and B and tangent to a given circle (or line) S;

(b) passing through a given point A and tangent to two given circles (or two lines, or a circle and a line) S_1 and S_2.

See also problems 13(a), (b) in NML 21 and problem 36(a). Problems 24(a), (b) are generalizations of problems 21(a), (b).

22. Draw a circle passing through two given points A and B and

(a) perpendicular to a given circle (or line) S;

(b) intersecting a given circle S in diametrically opposite points.

Problem 24(a) is a generalization of problem 22(a).

23. Draw a circle passing through a given point A and

(a) perpendicular to two given circles (or two lines, or a line and a circle) S_1 and S_2;

(b) perpendicular to a given circle (or line) S_1 and intersecting another given circle S_2 in two diametrically opposite points;

(c) intersecting two given circles S_1 and S_2 in two diametrically opposite points.

See also problems 37(a)–(c) in Section 3. Problem 24(b) is a generalization of problem 23(a).

24. Draw a circle

(a) passing through two given points A and B and cutting a given circle (or line) S at a given angle α;

(b) passing through a given point A and cutting two given circles (or two lines, or a line and a circle) S_1 and S_2 at given angles α and β.

25. Given three circles (or three lines, or two circles and a line, or two lines and a circle) S_1, S_2, and S_3. Draw a circle perpendicular to S_1 and S_2 and

(a) perpendicular to S_3;

(b) tangent to S_3;

(c) cutting S_3 at a given angle α.

See also problems 36(b) and 38(a) in Section 3. Problem 27(a) below is a generalization of problems 25(a), (b), (c).

26. Given three circles (or three lines, or two circles and a line, or two lines and a circle) S_1, S_2, and S_3. Draw a circle tangent to S_1 and S_2 and

(a) tangent to S_3 (the problem of Apollonius);

(b) cutting S_3 at a given angle α.

Problem 27(a) below is a generalization of problems 26(a), (b).

The problem of Apollonius is the problem of drawing a circle tangent to three given circles S_1, S_2, and S_3. There are many solutions of this famous problem some of which are given in this book (see the two solutions of problem 26(a), the solution of problem 61, and the two solutions of problem 74 in Section 5). Problems in which some (or all) of the circles S_1, S_2, and S_3 are replaced by points ("circles of zero radius") or by lines ("circles of infinite radius") are viewed by some as limiting versions of the problem of Apollonius; then "tangency" of a circle and a point means that the circle passes through the point. This broader viewpoint leads to the following ten variants of the problem of Apollonius:

Draw a circle that

1. passes through three given points;
2. passes through two given points and is tangent to a given line;
3. passes through two given points and is tangent to a given circle;
4. passes through a given point and is tangent to two given lines;
5. passes through a given point and is tangent to a given line and a given circle;
6. passes through a given point and is tangent to two given circles;
7. is tangent to three given lines;
8. is tangent to two given lines and a given circle;
9. is tangent to a given line and to two given circles;
10. is tangent to three given circles.

Of these ten problems two, namely problems (1) and (7), are included in the curriculum of elementary schools.[3] Problems (2) and (4) are involved in problems 13(a) and (b) in NML 21. Problem (8) is involved in problems 13(c) and 22 in NML 21 and in problem 60 in Section 5; problems (3), (5), and (6) are involved in problems 21(a) and (b). Finally, problems 9 and 10 (the Apollonius theorem proper) are involved in problem 26(a).

27. Given three circles S_1, S_2, and S_3. Draw a circle S such that

(a) the angles S forms with S_1, S_2, and S_3 have given magnitudes α, β, and γ;

(b) the segments of the common tangents of S and S_1, S and S_2, and S and S_3 have given lengths a, b, and c.

See also problems 75(a) and (b) in Section 5.

Using the properties of reflection in a circle it is easy to answer the question of which of the constructions of elementary geometry can be carried out using the compass alone. The usual assumption in construction problems is that one can use both ruler and compass. In some cases we can dispense with the compass. For relevant examples see problems 3(a),(b); 18(a),(b),(c),(d); and 32(b) in Sections 1 and 2 of NML 24. In Section 5 of NML 24 we showed that all constructions that can be carried out with ruler and compass can be carried out with ruler alone provided that we are given a circle with known center in the "drawing plane" (see pp. 100–101 in NML 24, where we give a more accurate formulation of this assertion).

Even more striking is the result dealing with constructions with compass alone. It turns out that using just a compass (and no ruler!) one can carry out all constructions which can be carried out using ruler and compass.[4] We prove this in problems 28–32 below. (This result was first proved by the 17th-century Dutch mathematician George Mohr and reproved—when Mohr's work was forgotten—by the 18th-century Italian mathematician Lorenzo Mascheroni.)

28. Given a segment AB in the plane. Double AB using just a compass (that is, extend AB beyond B and find on the extension a point C such that $AC = 2AB$).

The solution of problem 28 implies that using a compass alone we can obtain a segment that is n times longer than a given segment AB. In other words, we can find on the extension of AB beyond B a point C such that $AC = nAB$. For example, to triple AB we first find C' on the extension of AB beyond B such that $AC' = 2AB$ and then C on the extension of BC' beyond C' such that $BC = 2BC'$. Then, clearly, $AC = 3AB$ (Figure 40).

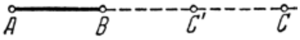


FIGURE 40

29. A circle Σ with known center O and a point A are given in a plane. Using just a compass find the point A' symmetric to A with respect to Σ.

30. Given a circle S in a plane. Find its center using just a compass.

31. Using just a compass, draw a circle passing through three given (non-collinear) points A, B, and C.

32. (a) A circle Σ and a line l are given in a plane (l may be given by just two of its points A and B). Using just a compass, find the circle l' symmetric to l with respect to Σ.

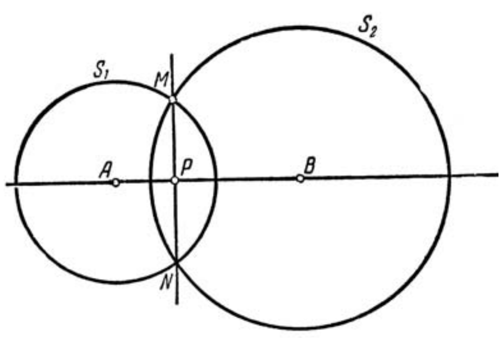
(b) Given two circles Σ and S in a plane. Using just a compass, find the circle (or line) S' symmetric to S with respect to Σ.

The results of problems 29, 31, and 32(a) and (b) justify the claim that every construction problem which can be carried out using ruler and compass can be carried out using compass alone. In fact, suppose that we can solve a certain construction problem using ruler and compass. Let F be the drawing associated with this construction. F consists of certain circles and lines. If we invert F in a circle Σ whose center O is not a point of any

of the circles or lines of F, then we obtain a new drawing F' made up of just circles. We will show that F' can be constructed with just a compass.

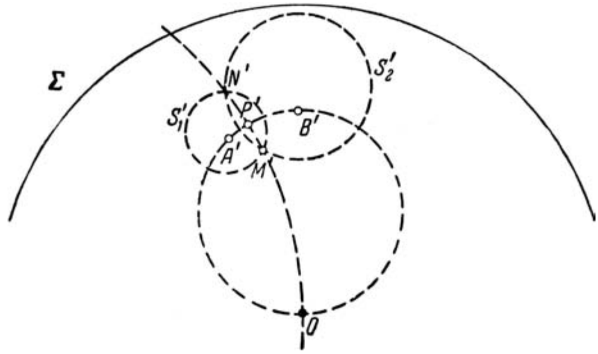
The process of construction of F involves the successive construction of certain lines and circles. We transfer the given points of F to F' (in view of problem 29 this can be done with compass alone). Then we construct F' in the same order in which F was constructed. Each line l of F passes through certain points A and B that are already in F. The circle l' in F', which corresponds to l in F, passes through points A' and B', which correspond to A and B, and through the center O of Σ. In the process of construction of F' the construction of A' and B' precedes the drawing of l'. This being so, l' can be constructed with just a compass (see problem 31). Each circle S of F is constructed using its known center Q and radius MN.[5] In the process of "sequential" construction of F', the points Q', M', and N', corresponding to the points Q, M, and N, are constructed before the construction of the circle S' corresponding to the circle S of F. In view of the results in problems 29 and 32(b), we can construct the points Q, M, and N of F, then the circle S, and, finally, the required circle S' of F' using a compass alone. In other words, we can construct F', that is, reconstruct the figure F' obtained by a given inversion from the required figure F (each of F and F' may consist of just one point). But then, the reconstruction of F presents no difficulties (see, however, Note 4).

To clarify the general method just described we consider in detail two examples of its use. Figure 41a shows the familiar construction of the perpendicular from a point M to a line AB. This construction presupposes the use of ruler and compass. To obtain the foot P of the perpendicular



(a)

FIGURE 41

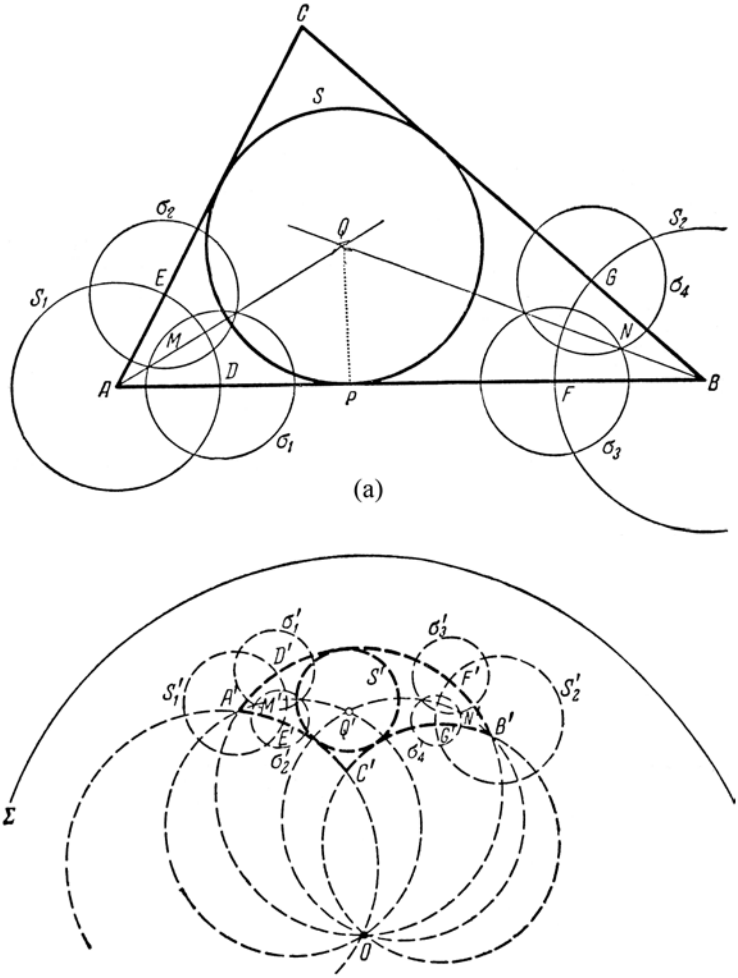


(b)

FIGURE 41

with compass alone we can proceed as follows. We consider the Figure 41b symmetric to Figure 41a with respect to the circle Σ with center O. We construct the points A', B', and M' symmetric to the points A, B, and M with respect to Σ (problem 29). Then we construct circles S_1 and S_2 with centers A and B and radii AM and BM; to these circles there correspond circles S_1' and S_2' which intersect at M' and N'. (The circles S_1' and S_2' are easy to obtain; see problem 32(b).[6]) The point P' in Figure 41b can be obtained as a point of intersection of the circles $A'B'O$ and $M'N'O$ (see problem 31), and the required point P as the point symmetric to P' with respect to Σ (problem 29). [It is clear that using a compass alone we can construct as many points of the perpendicular MP, symmetric to points of the circle $M'N'O$, as we wish. But of course we cannot construct *all* the points of the segment MP.]

As our second example we consider the problem of constructing the circle S inscribed in a given triangle ABC (Figure 42a; the triangle can be thought of as given by just its vertices). We find the points A', B', C' symmetric to the vertices of the triangle with respect to a circle Σ with center O (Figure 42b). We construct the circles $A'B'O$, $A'C'O$, and $B'C'O$ (problem 31) and the circles S_1' and S_2' corresponding to arbitrary circles S_1 and S_2 with centers A and B (problem 32(b)). To the points of intersection of the circle S_1' with the circles $A'B'O$ and $A'C'O$, and of the circle S_2' with the circles $A'B'O$ and $B'C'O$, there correspond the points D and E, and F and G in Figure 42a (problem 29); to the circles σ_1 and σ_2, σ_3 and σ_4, with the same radii and centers D and E, F and G, there correspond the circles σ_1' and σ_2', σ_3' and σ_4' in Figure 42b (problem 32(b)). We denote a point of



(a)

(b)

FIGURE 42

intersection of σ_1' and σ_2' by M', a point of intersection of σ_3' and σ_4' by N', and a point of intersection of the circles $A'M'O$ and $B'N'O$ (problem 31) by Q'; the point Q' is symmetric to the center Q of the circle inscribed in the triangle ABC. All that remains is to drop from Q the perpendicular QP to the side AB of the triangle (the problem of constructing the foot P of this perpendicular using just a compass was considered above) and then to draw the circle with center Q and radius QP.

Notes to Section 2

[1] One can also use the fact that the endpoints M' and N' of the diameter $M'N'$ of the circle S' passing through O are symmetric with respect to Σ to the endpoints M and N of the diameter of S passing through O (see Figure 12).

[2] One can also take for the circle S any circle whose center O is on the radical axis of the circles S_1 and S_2 and whose radius is equal to the length of the tangent from O to S_1 or S_2 (see p. 49).

[3] It is safe to say that problem 7 is seldom discussed in elementary school. This being so, it is useful to note that this problem has four solutions if the three lines form a triangle (the inscribed and the three escribed circles of that triangle), two solutions if two of the lines are parallel and the third one intersects them, and one solution if all three lines are either parallel or concurrent.

[4] This statement calls for a refinement analogous to the refinement made in connection with the discussion of constructions with ruler alone (see pp. 100–101 in NML 24). If we are to construct (part of) a line or rectilinear figure (say, a triangle), then, clearly, this cannot be done with compass alone. However, we can obtain as many points of the required lines as desired, as well as all points of intersection of the required lines and circles (for example, all vertices of a required triangle).

[5] The most common case is the case when the point M coincides with Q.

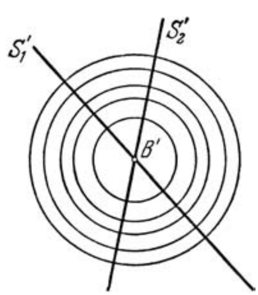
[6] It is simpler to find at once the point N' symmetric to Σ with respect to the point N of intersection of S_1 and S_2.

3
Pencils of circles. The radical axis of two circles

The proof of the important Theorem 2 in Section 1 (p. 22) involved circles perpendicular to two given circles. Such circles play an important role in an issue of fundamental significance which we are about to discuss in detail.

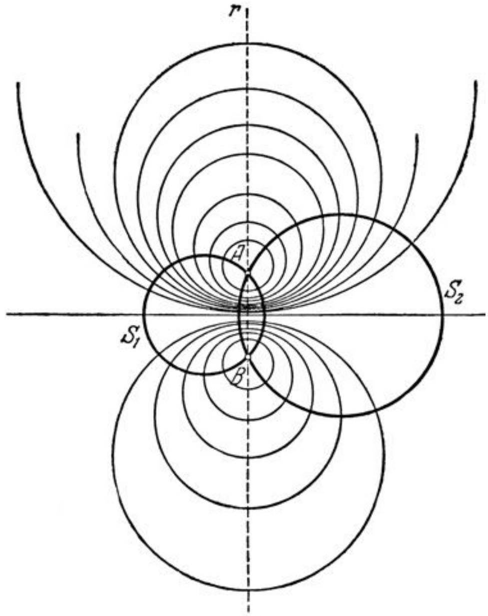
The set of all circles (and lines) perpendicular to two given circles S_1 and S_2 (or to a circle and a line, or to two lines) is called a pencil of circles.[1] Depending on the mutual disposition of S_1 and S_2 we distinguish three types of pencils of circles.

1^0. If the circles S_1 and S_2 intersect each other, we can use an inversion to transform them into two intersecting lines S_1' and S_2' (see Theorem 2 in Section 2); then the pencil of circles perpendicular to S_1 and S_2 goes over to the pencil of circles perpendicular to S_1' and S_2'. But if a circle is perpendicular to a line, then its center must lie on this line; hence all circles perpendicular to S_1' and S_2' are concentric with common center at the point B' of intersection of S_1' and S_2' (Figure 43a). Hence *a pencil of*



(a)

FIGURE 43

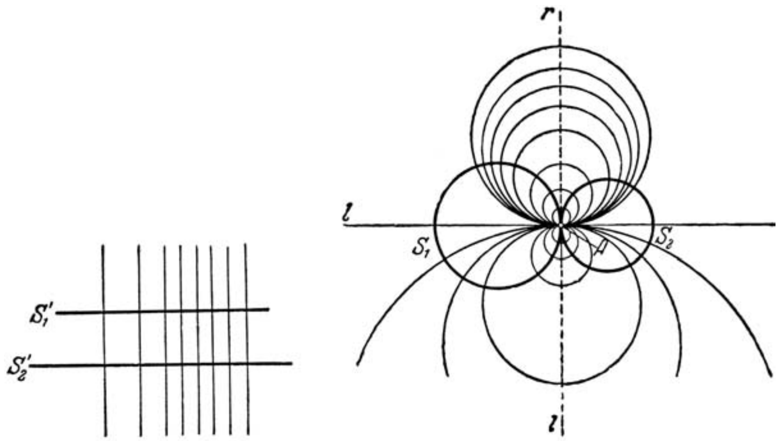


(b)

FIGURE 43

circles perpendicular to two intersecting circles S_1 and S_2 can be taken by inversion to a pencil of concentric circles; but then no two of its circles intersect one another (Figure 43b). The image of that one of the concentric circles which passes through the center of inversion A is a line of the pencil. Since the concentric circles include small circles arbitrarily close to B', it follows that among the circles of our pencil are small circles arbitrarily close to B, the point of intersection of S_1 and S_2. This being so, B itself is included, by convention, in the pencil (and similarly B', the common center of the totality of concentric circles, is included in that set). Since the concentric circles include arbitrarily large ones, it follows that our pencil includes arbitrarily small circles close to the center of inversion A. This being so, the point A itself, the second point of intersection of S_1 and S_2, is included, by convention, in the pencil.

2^0. It is possible to take two tangent circles S_1 and S_2 to two parallel lines S_1' and S_2'. If a circle is to be perpendicular to two parallel lines, it is necessary that its center should be a point of each of these two lines. Since this is impossible, it follows that the totality of "circles" perpendicular to



(a) (b)

FIGURE 44

S_1' and S_2' consists of the lines perpendicular to S_1' and S_2' (Figure 44a). Going back by inversion to S_1 and S_2 we conclude that *a pencil of circles perpendicular to two tangent circles S_1 and S_2 can be taken by inversion to a totality of parallel lines*. This pencil consists of circles passing through the point A of tangency of S_1 and S_2 (the center of inversion) and tangent at A to the line l perpendicular to S_1 and S_2; l also belongs to the pencil (Figure 44b). Sometimes the point A is included, by convention, in the pencil.

3⁰. Two nonintersecting circles S_1 and S_2 can be taken by inversion to two concentric circles S_1' and S_2'. If circles S and Σ that intersect at M are mutually perpendicular (Figure 45), then the tangent to Σ at M is perpendicular to the tangent to S at M, and therefore passes through the

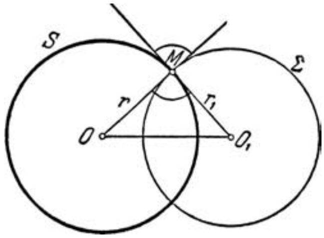


FIGURE 45

center O of S; analogously, the tangent to S passes through the center O_1 of Σ. It follows that the triangle OMO_1 is a right triangle. This implies that the sum of the squares of the radii of S and Σ is equal to the square of the distance between their centers. Hence, obviously, *two circles with different radii and the same center (concentric circles) cannot be perpendicular to the same circle*. Thus the pencil of circles perpendicular to S_1' and S_2' contains no circles; it consists of lines passing through the common center P' of S_1' and S_2' (Figure 46a). From all this we can draw the conclusion that *the pencil of circles perpendicular to two nonintersecting circles S_1 and S_2 can be taken by inversion to the totality of lines intersecting in a single point*; the pencil consists of all circles passing through two fixed points, P (which corresponds under inversion to P') and Q (the center of inversion (see Figure 46b)). The line joining P and Q also belongs to the pencil.[2]

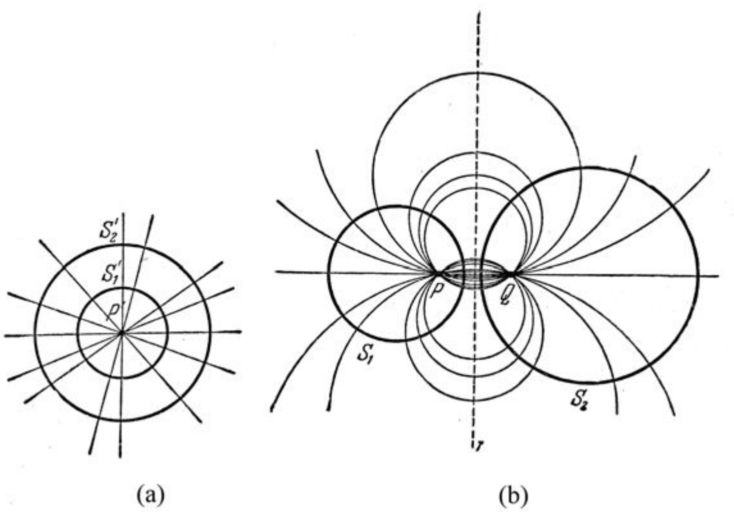


FIGURE 46

(*Translator's remark.* Each of the "totalities" in $\mathbf{1^0}$, $\mathbf{2^0}$, and $\mathbf{3^0}$ is a pencil, but it is perhaps useful to reserve the term "pencil" for the pencils whose structure is being determined.)

Figures 43, 44, and 46 enable us to perceive certain common properties of all three types of pencils. For example, regardless of the type of pencil, through each point in the plane there passes a circle of the pencil and, in general, that circle is unique. The exceptions are: the point A of the pencil

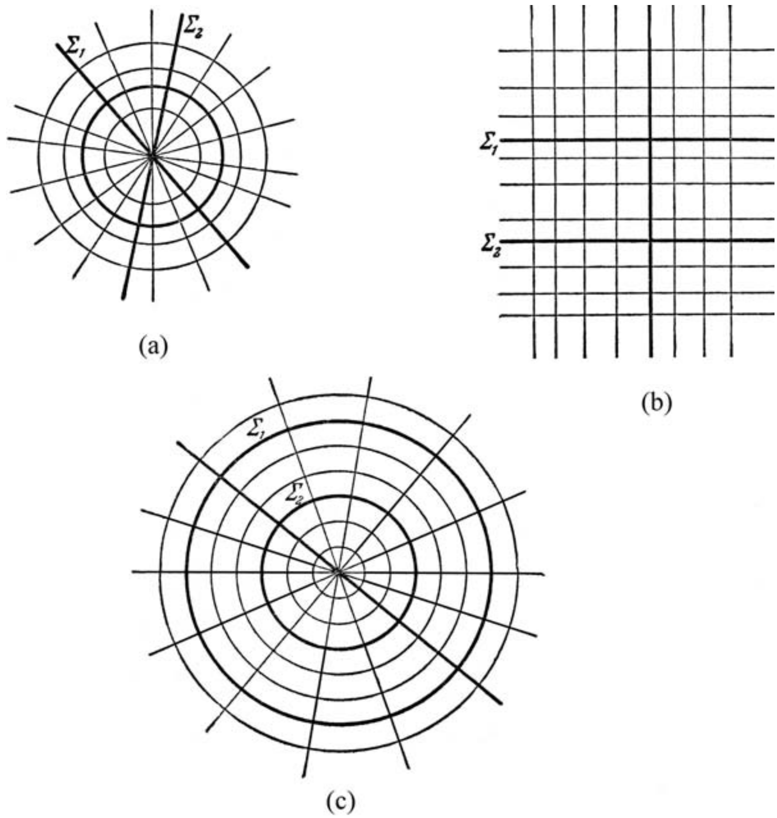
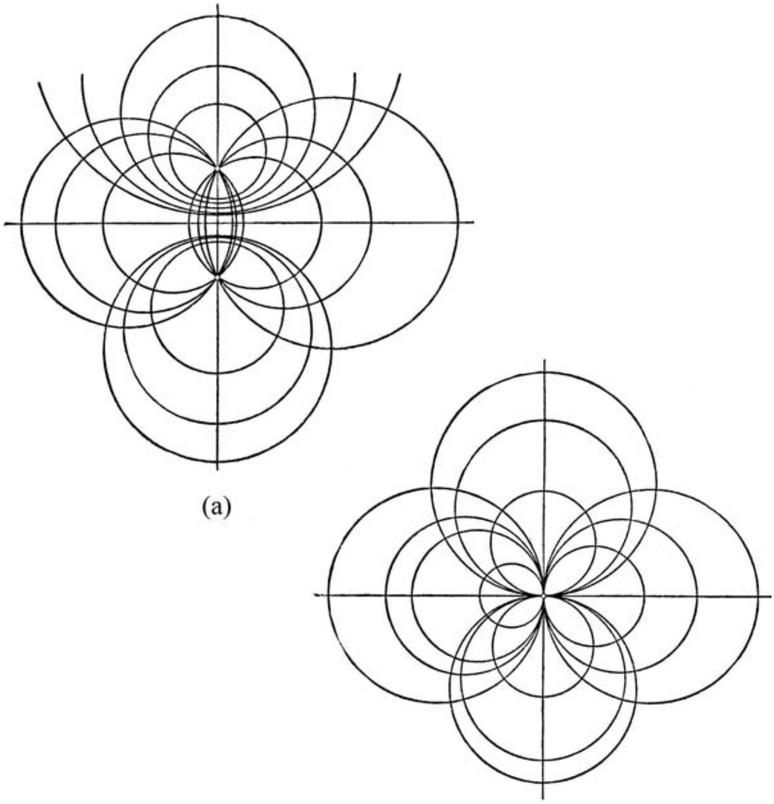


FIGURE 47

of tangent circles (Figure 44b) and the points P and Q of the pencil of intersecting circles (Figure 46b). In each of these two cases the number of circles passing through the relevant points is infinite.

We note also that the circles perpendicular to two circles Σ_1 and Σ_2 of a pencil are perpendicular to all circles of that pencil. This is obvious if Σ_1 and Σ_2 are lines (Figures 47a and 47b) or two concentric circles (Figure 47c). From this and from Theorem 2 in Section 1 it follows that this assertion is true in all cases. This means that we can speak of mutually perpendicular pencils of circles. It is obvious that a pencil of circles perpendicular to a pencil of intersecting circles consists of circles without common points, and conversely (Figure 48a); and that a pencil of circles perpendicular to a pencil of tangent circles consists of tangent circles (Figure 48b).



(a) (b)

FIGURE 48

For what follows it is essential to note that *the centers of the circles of a pencil lie on a single line* r. This is obvious in the case of a pencil of tangent circles (in that case r passes through A and is perpendicular to l; see Figure 44b) as well as in the case of intersecting circles (in that case r is perpendicular to PQ and passes through its midpoint; Figure 46b). In the case of a pencil of nonintersecting circles (Figure 43b) our assertion follows from the fact that the pencil can be taken by inversion to a pencil of concentric circles, and, therefore, all circles of the pencil are centrally similar with the same center of similarity A (but with different coefficients of similarity; see the proof of property B_4 of inversion on p. 9) to the circles with the same center B'; this means that their centers lie on the single line AB' (coincident with AB, for B' lies on the line AB).

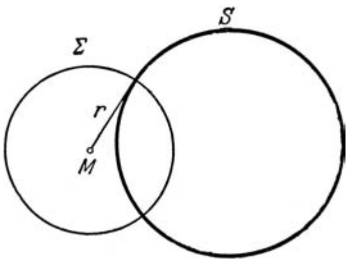
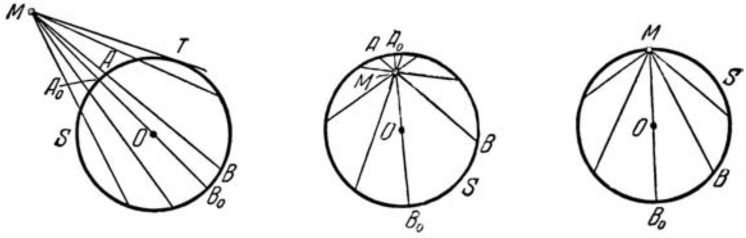


FIGURE 49

The latter observation implies an interesting consequence. If a circle Σ with center M and radius r is perpendicular to a circle S, then the tangent from M to S has length r (Figure 49). It follows that if a circle with center M is perpendicular to two circles S_1 and S_2, then the tangents from M to these circles have the same length; conversely, if the tangents MT_1 and MT_2 from M to S_1 and S_2 have the same length, then the circle with center M and radius $MT_1 = MT_2$ is perpendicular to S_1 and S_2. Hence the fact that the centers of the circles of a pencil lie on a single line implies that *the points from which one can draw tangents of equal length to two circles S_1 and S_2 lie on a single line*. This line is called the radical axis of S_1 and S_2.

Since the notion of a radical axis is very important, we approach it from yet another side. In a first course in geometry the length of the tangent from a point M to a circle S appears in the following theorem:

Theorem 1a. *Let M be a point in the exterior of a circle S. Let A and B be points of S in which a secant from M cuts S. Then the product $MA \cdot MB$ depends only on M and S and is independent of the choice of secant; this product is equal to the square of the tangent from M to S* (Figure 50a).



(a) (b) (c)

FIGURE 50

Theorem 1a is analogous to the following theorem:

Theorem 1b. *Let M be a point in the interior of a circle S. Let A and B be the endpoints of a chord of S passing through M. Then the product $MA \cdot MB$ depends only on M and S and is independent of the choice of chord* (Figure 50b).

Theorems 1a and 1b show that if we draw lines through a point M that cut a circle S at points A and B, then the product $MA \cdot MB$ depends only on the choice of point and circle and not on the choice of line. According to our definition of the signs of segments (see Section 1), this product is positive if M is in the exterior of S, negative if M is in the interior of S, and zero if M is on S (Figure 50c). The signed number $MA \cdot MB$ is called *the power of the point M with respect to the circle S.*

Let d be the distance from M to the center O of S and r the radius of S. Consider the secant MA_0B_0 through the center O in Figures 50a and 50b. In Figure 50a we have: $MA_0 = d - r$ and $MB_0 = d + r$; and in Figure 50b we have: $MA_0 = r - d$ and $MB_0 = d + r$. Keeping in mind our sign convention we see that in all cases *the power of M with respect to S is $d^2 - r^2$.*

The points from which we can draw equal tangents to circles S_1 and S_2 have the property that their powers with respect to S_1 and S_2 have the same value. This makes it possible to reformulate the fact that all such points lie on the same line as follows: *All points exterior to given circles S_1 and S_2 and having the property that their powers with respect to S_1 and S_2 have the same value lie on the same line.* Next we prove that *the locus of points whose powers with respect to two circles S_1 and S_2 have the same value is a line.*[3] This line is called the radical axis of S_1 and S_2.

Let M be a point of the required locus and let d_1 and d_2 be its distances from the centers O_1 and O_2 of the given circles S_1 and S_2 with radii r_1 and r_2 (Figures 51a and 51b). We have the equality

$$d_1^2 - r_1^2 = d_2^2 - r_2^2.$$

Drop the perpendicular MP from M to the line of centers O_1O_2. Clearly, $d_1^2 = MP^2 + PO_1^2$, $d_2^2 = MP^2 + PO_2^2$. Hence

$$MP^2 + PO_1^2 - r_1^2 = MP^2 + PO_2^2 - r_2^2.$$

It follows that

$$PO_1^2 - PO_2^2 = r_1^2 - r_2^2,$$

or

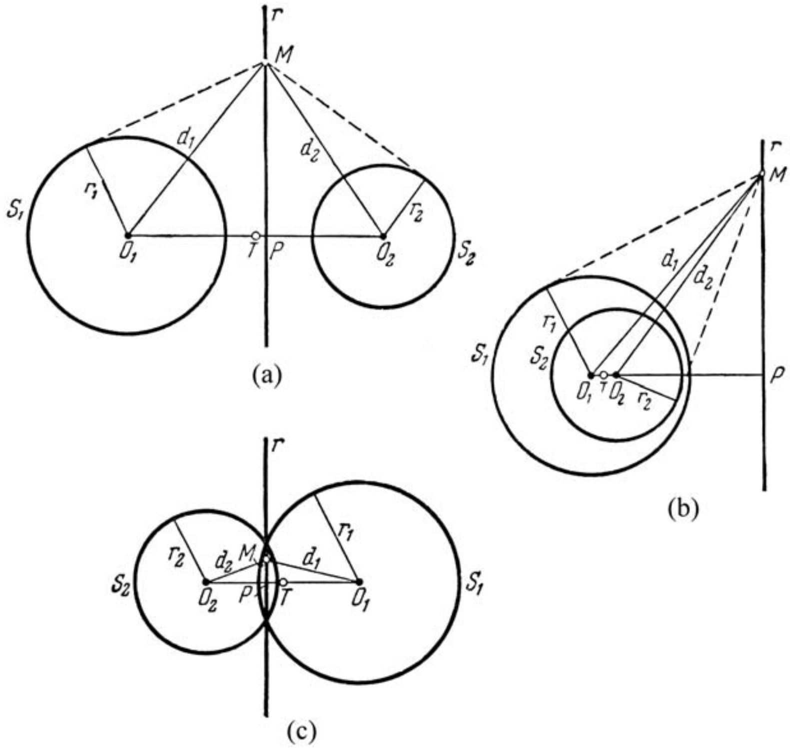


FIGURE 51

$$(PO_1 + PO_2)(PO_1 - PO_2) = r_1^2 - r_2^2.$$

Now let $r_1 \geq r_2$. Then $PO_1 \geq PO_2$, that is, P is not farther from O_2 than from O_1; in other words, if T is the midpoint of the segment O_1O_2, then we can say that P is on the same side of T as O_2 (the center of the smaller circle). If P lies between O_1 and O_2 (Figures 51a, 51c, and 51e), then

$$PO_1 + PO_2 = O_1O_2$$

$$PO_1 - PO_2 = (PT + TO_1) - (TO_2 - PT) = 2PT;$$

if P lies outside the segment O_1O_2 (Figures 51b, 51d, and 51f), then

$$PO_1 - PO_2 = O_1O_2,$$

$$PO_1 + PO_2 = (PT + TO_1) + (PT - TO_2) = 2PT.$$

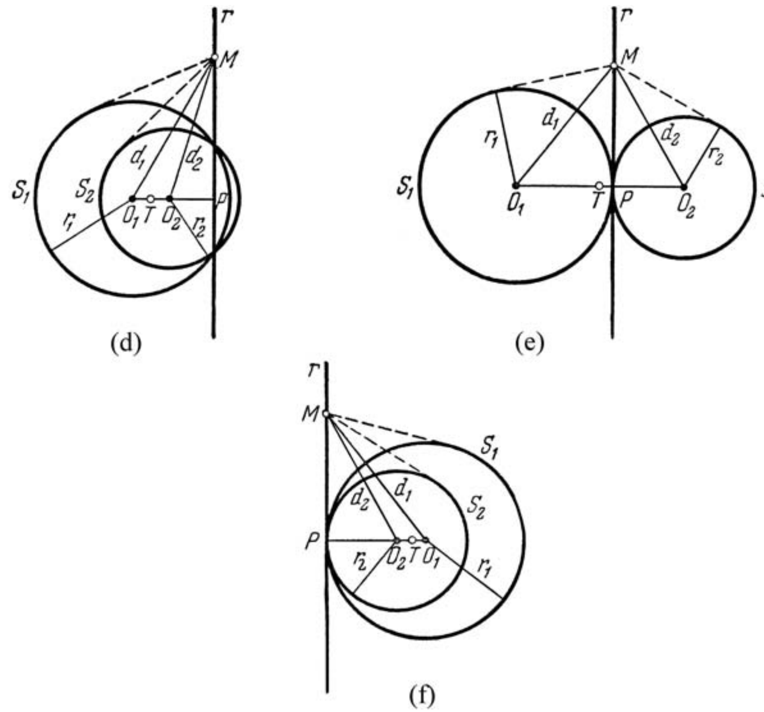


FIGURE 51

Thus in all cases $PO_1^2 - PO_2^2 = 2O_1O_2 \cdot TP$, and, ultimately, $2O_1O_2 \cdot TP = r_1^2 - r_2^2$.

We conclude that P does not depend on the choice of the point M of the required locus, which means that the latter is a line perpendicular to the line O_1O_2 determined by the centers O_1 and O_2.

It is clear that if the circles S_1 and S_2 have no points in common, then their radical axis intersects neither of them (Figures 51a and 51b); if S_1 and S_2 have two points in common, then their radical axis passes through these points (which have power zero with respect to S_1 and S_2 (see Figures 51c and 51d)); if S_1 and S_2 are tangent to one another, then their radical axis passes through their point of tangency (see Figures 51e and 51f).

Using the formula

$$TP = \frac{r_1^2 - r_2^2}{2O_1O_2}$$

we can easily establish the following. If the circles S_1 and S_2 are exterior to

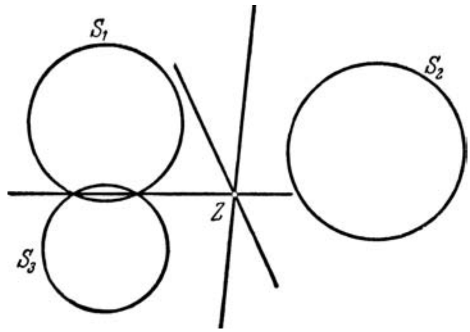


FIGURE 52

each other, then their radical axis passes between them (Figure 51a). If S_2 is in the interior of S_1, then their radical axis is outside S_1 and S_2 (Figure 51b). If S_1 and S_2 are congruent ($r_1 = r_2$), then their radical axis coincides with their axis of symmetry. If S_1 and S_2 are concentric circles ($O_1O_2 = 0$) then they have no radical axis.[4]

Let S_1, S_2, and S_3 be three given circles. If their centers are not collinear, then the radical axes of the three pairs of these circles (perpendicular to the lines joining their centers) are not parallel. If Z is the point of intersection of the radical axes of the pairs S_1, S_2 and S_1, S_3, then the power of Z with respect to the pair S_1, S_2 is the same as its power with respect to the pair S_1, S_3. But then this is also the power of Z with respect to the pair S_2, S_3, that is, Z lies on the radical axis of the pair S_2, S_3. In other words, *the radical axes of the three pairs of circles obtained from three circles S_1, S_2, and S_3 whose centers are not collinear meet in a point* (Figure 52). This point is called the *radical center* of the three circles.

This discussion implies a simple method for the construction of the radical axis r of two nonintersecting circles (the radical axis of two intersecting circles coincides with the line determined by their common chord). We construct an auxiliary circle S intersecting S_1 and S_2 (Figure 53). The radical axes of the pairs S_1, S and S_2, S are lines determined by their common chords. Hence the point M of intersection of the lines determined by these common chords lies on r. One way of obtaining r is to drop a perpendicular from M to the line of centers of S_1, S, and S_2. Another way is to introduce a second auxiliary circle S' and to obtain in this way another point M' of r.

We note that in the preceding discussion we could have assumed that some (or all) of the circles under consideration had zero radius, that is, that

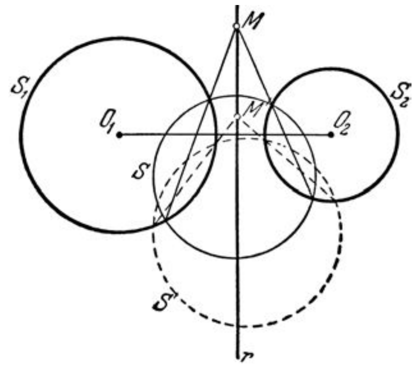
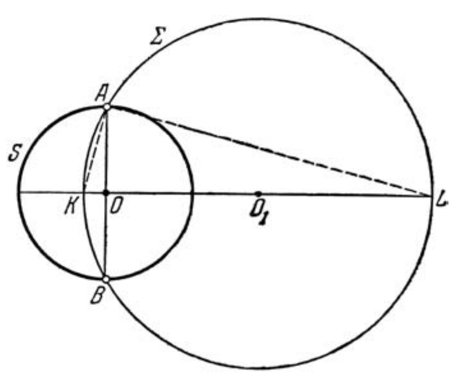


FIGURE 53

they were points. In particular, the power of a point M with respect to a point S is the square of the distance MS, the radical axis of two points is their axis of symmetry, and the radical center of three points is the center of the circle passing through these points.

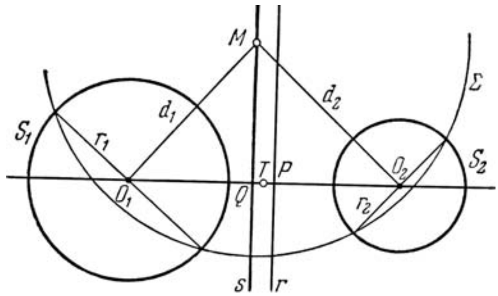
We note the following proposition, analogous to the theorem on the centers of circles perpendicular to two given circles (see p. 49): *the locus of the centers of circles intersecting two given circles S_1 and S_2 in diametrically opposite points is the line s perpendicular to the line of centers of S_1 and S_2* (and thus parallel to their radical axis). In fact, if a circle Σ intersects a circle S in diametrically opposite points A and B and their line of centers OO_1 intersects Σ in points K and L (Figure 54a), then the right triangle KAL yields the equality

$$KO \cdot OL = OA^2,$$



(a)

FIGURE 54



(b)

FIGURE 54

or, denoting the radii $O_1K = O_1L$ and OA by R and r and the distance OO_1 between their centers by d, the equality

$$(R-d)(R+d) = r^2, \quad R^2 = d^2 + r^2.$$

Hence if M is the center of the circle Σ that intersects the diameters of circles S_1 and S_2 with centers O_1 and O_2 and radii r_1 and r_2 (Figure 54b), then

$$d_1^2 + r_1^2 = d_2^2 + r_2^2, \quad d_2^2 - d_1^2 = r_1^2 - r_2^2;$$

here $d_1 = MO_1$ and $d_2 = MO_2$. It follows that if $r_1 \geq r_2$, then $d_1 \leq d_2$. Also, transforming the second of the displayed equalities, we obtain the equality

$$2O_1O_2 \cdot TQ = r_1^2 - r_2^2,$$

where Q is the foot of the perpendicular from M to O_1O_2 and T is the midpoint of the segment O_1O_2 (see pp. 50–52). It follows that the required locus of points M is the line s symmetric to the radical axis r of the circles S_1 and S_2 with respect to the midpoint T of the segment O_1O_2.

We will now find the locus of circles Σ such that two given circles S_1 and S_2 intersect Σ in diametrically opposite points (Figure 55). It is clear that the center of Σ must lie in the interior of both S_1 and S_2 (see Figure 54a). Hence such circles exist only if S_1 and S_2 intersect one another. Further, let r_1, r_2, and R be the radii of S_1, S_2, and Σ, and let d_1 and d_2 be the distances from M to the centers of S_1 and S_2. Since S_1 and S_2 intersect Σ at diametrically opposite points, it follows that

$$r_1^2 = R^2 + d_1^2, \quad \text{and} \quad r_2^2 = R^2 + d_2^2,$$

whence

$$R^2 = r_1^2 - d_1^2 = r_2^2 - d_2^2; \quad d_1^2 - r_1^2 = d_2^2 - r_2^2.$$

But this equality shows that the powers of M with respect to S_1 and S_2 are equal. This means that M is a point on the radical axis of S_1 and S_2. Consequently, *the locus of centers of circles Σ such that S_1 and S_2 intersect Σ in diametrically opposite points is the segment of the radical axis r of S_1 and S_2 in the interior of S_1 and S_2, that is, a common chord of S_1 and S_2* (the part of the radical axis in

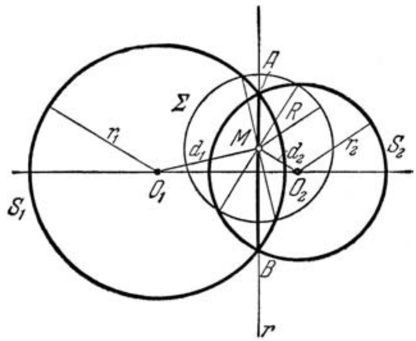


FIGURE 55

the exterior of S_1 and S_2 is the locus of centers of circles Σ such that S_1 and S_2 cut Σ at a right angle).

We consider once more two mutually perpendicular pencils of circles (see Figures 48a and 48b). We noted that the radical axis of any two circles S_1 and S_2 of one of these pencils is the line of centers of the pencil perpendicular to it (see p. 49), which implies that *any two circles of the pencil have the same radical axis*. Conversely, the totality of circles with the same radical axis is a pencil of circles (perpendicular to the pencil of circles perpendicular to any two circles S_1 and S_2 from this set).

Sometimes one defines a pencil of circles as the totality of circles any two of which have the same radical axis.[5] But this definition is less appropriate than the one we gave earlier, for it does not include totalities of concentric circles (Figure 43a), of parallel lines (Figure 44a), and of lines intersecting in a fixed point (Figure 46a).[6]

The analogue of a pencil of circles—a totality of circles any two of which have the same radical axis—is a bundle of circles—a totality of circles any three of which have the same radical center Z. [By convention we apply the term "bundle" to a totality of lines in the plane.] We distinguish three kinds of bundles, depending on whether Z is in the exterior of the circles of the bundle, on these circles, or in their interior.[7] A bundle of the second kind consists of all circles passing through a fixed point Z. It can be shown that a bundle of the first or third kind consists of a totality of circles that go over to themselves under a definite fixed inversion I (of positive power for a bundle of the first kind and of negative power for a bundle of the third kind). A bundle of the first kind can also be defined as a totality of circles (and lines) perpendicular to some circle Σ, and a bundle of the third kind can also be defined as a totality of circles (and lines) that intersect a circle Σ in two diametrically opposite points. For proofs of these assertions see problems 273–291 in the "Collection of Geometric Problems" by B. N. Delone and O. K. Zhitomirskiĭ, Gostekhizdat, Moscow-Leningrad, 1950. (Russian)

33. Show that the common chords of three pairwise intersecting circles meet in a point (Figure 56) or are parallel.

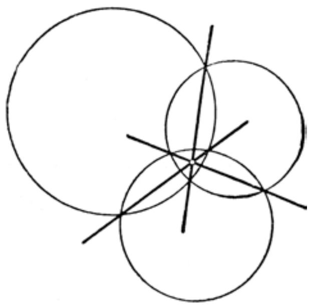


FIGURE 56

34. Show that the common chord of two intersecting circles halves the segment of their common outer tangent whose endpoints are the points of tangency.

35. Let S be a given circle and M a given point in its exterior. Let Σ be a variable circle through M intersecting S in points A and B. Find the locus of points of intersection of the line AB and the tangent to Σ at M.

36. Construct a circle

(a) passing through two given points A and B and tangent to a given circle (or line) S;

(b) perpendicular to two given circles S_1 and S_2 and tangent to a given circle (or line) S.

See also problem 13(b) in NML 21 and problems 21(a), (b) in this book.

37. Construct a circle passing through a given point M and

(a) perpendicular to two given circles S_1 and S_2;

(b) perpendicular to a given circle S_1 and intersecting a second given circle S_2 in diametrically opposite points;

(c) intersecting two given circles S_1 and S_2 in diametrically opposite points.

See also problems 23(a), (b), and (c) in this book.

38. Given three circles S_1, S_2, and S_3 construct a circle S

(a) perpendicular to S_1, S_2, and S_3;

(b) intersecting S_1, S_2, and S_3 in diametrically opposite points;

(c) such that S_1, S_2, and S_3 intersect S in diametrically opposite points.

See also problem 25(a) in this book.

39. Given two circles S_1 and S_2. Find the locus of points M such that

(a) the difference of the squares of the lengths of the tangents from M to S_1 and S_2 has a given value a;

(b) the ratio of the lengths of the tangents from M to S_1 and S_2 has a given value k.

See also problem 64 in NML 21 and problem 49 in this book.

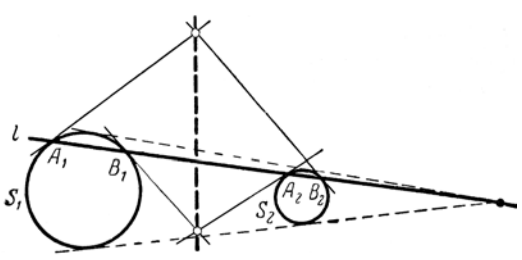
In problems 39(a) and (b) we can replace one of the two circles S_1 and S_2, or both of them, by points ("circles with zero radius"); here, by the length of the tangent from M to S we mean the distance between these points. Thus these problems are generalizations of the following simpler problems:

(a) find the locus of points such that the difference of the squares of the distances from these points to two given points is constant;

(b) find the locus of points such that the ratio of the distances from these points to two given points is constant.

40. Given two circles S_1 and S_2 and a line l that intersects S_1 in points A_1 and B_1 and S_2 in points A_2 and B_2. Show that

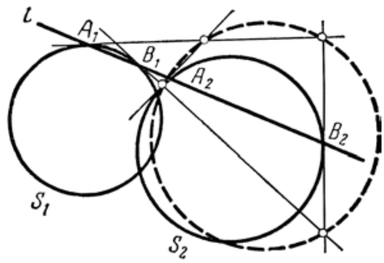
(a) if l passes through the center of similarity of S_1 and S_2, then the tangents to S_1 at A_1 and B_1 intersect the tangents to S_2 at A_2 and B_2 in two points on the radical axis of S_1 and S_2 (Figure 57a);



(a)

FIGURE 57

(b) if l does not pass through the center of similarity of S_1 and S_2, then the tangents to S_1 at A_1 and B_1 intersect the tangents to S_2 at A_2 and B_2 in four concyclic points (Figure 57b).



(b)

FIGURE 57

41. The theorem of Poncelet. Prove that if there is an n-gon inscribed in a given circle S and circumscribed about another given circle s, then there are infinitely many such n-gons; we can take any point on the larger circle S as a vertex of any such n-gon (Figure 58).

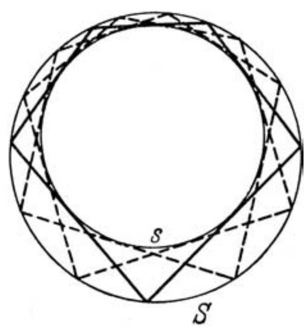


FIGURE 58

See also problems 8(a) and 9(b) in this book.

Notes to Section 3

[1] Strictly speaking, one should use the term "pencil of circles and lines." However, "lines" is usually omitted for brevity.

[2] In the literature, the pencil of circles in 1^0 (the pencil of nonintersecting circles) is called *hyperbolic*, the pencil of circles in 2^0 (the pencil of tangent circles) is called *parabolic*, and the pencil of circles in 3^0 (the pencil of intersecting circles) is called *elliptic*.

[3] If S_1 and S_2 have no common points, then all points whose powers with respect to S_1 and S_2 have the same value are exterior to S_1 and S_2; hence, in that case, our assertion can be viewed as proved.

[4] This is also a consequence of the fact that there are no circles perpendicular to each of two concentric circles (see 3^0 on p. 45).

One sometimes says that the radical axis of two concentric circles is "at infinity." Here we are guided by the same notions that made us introduce elements at infinity on p. 29.

[5] It is interesting to note that, with this definition of the term pencil, points which we often included by convention in pencils of type 1^0 and 2^0 are included quite naturally in these pencils; this is so because we can speak of the radical axis of a point and a circle as well as of the radical axis of two points. But now lines must be added to the circles of a pencil by convention (with the proviso that the radical axis of any two circles of the pencil also belongs to the pencil).

[6] An important distinction of the first definition of a pencil of circles is that it implies directly that *inversion takes a pencil of circles to a pencil of circles*; in this context the second definition is less appropriate.

[7] A bundle of the first kind is called hyperbolic; of the second kind—parabolic; and of the third kind—elliptic.

4
Inversion (concluding section)

In Section 1 we saw that the angle between two lines (or between a line and a circle, or between two circles) is unchanged by an inversion (see property C of inversion, p. 11). We will now show what effect an inversion has on the distance between two points. Let A and B be two points in the plane and let A' and B' be their images under the inversion with center O (different from A and B) and power k, which we take as positive for the sake of simplicity (Figure 59). The triangles OAB and $OB'A'$ are similar because $\angle AOB = \angle A'OB'$ and $\frac{OA}{OB} = \frac{OB'}{OA'}$ (for $OA \cdot OA' = OB \cdot OB' = k$). It follows that

$$\frac{AB}{A'B'} = \frac{OA}{OB'}, \quad \text{whence} \quad A'B' = AB \cdot \frac{OB'}{OA}.$$

Replacing OB' by $\frac{k}{OB}$ we obtain the required formula

$$A'B' = AB \frac{k}{OA \cdot OB}. \tag{$*$}$$

If k is negative, then we need only replace k in $(*)$ by $|k|$.

42. (a) Let $d_1, d_2, d_3, \ldots, d_n$ be the distances from a point M on the arc A_1A_n of the circle circumscribed about a regular n-gon $A_1A_2A_3 \ldots A_n$ to

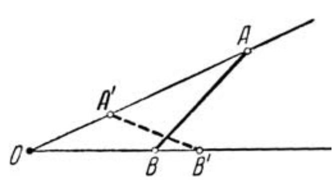


FIGURE 59

the vertices $A_1, A_2, A_3, \ldots, A_n$ of that n-gon. Show that

$$\frac{1}{d_1d_2}+\frac{1}{d_2d_3}+\frac{1}{d_3d_4}+\cdots+\frac{1}{d_{n-1}d_n}=\frac{1}{d_1d_n}.$$

[In particular, for $n = 3$ we obtain

$$\frac{1}{d_1d_2}+\frac{1}{d_2d_3}=\frac{1}{d_1d_3},$$

whence

$$d_3+d_1=d_2.$$

In other words, *the sum of the distances from a point on the circle circumscribed about an equilateral triangle to the two nearest vertices of that triangle is always equal to its distance from the third vertex.*]

(b) Show that if n is odd then, using the notations in problem (a), we have the equality

$$d_1+d_3+d_5+\cdots+d_n=d_2+d_4+\cdots+d_{n-1}.$$

[In particular, for $n = 3$ we obtain the same proposition as in problem (a).]

43. (a) Let $p_1, p_2, p_3, \ldots, p_{2n-1}, p_n$ be the distances from a point M on a circle S to the sides $A_1A_2, A_2A_3, \ldots, A_{2n-1}A_{2n}, A_{2n}A_1$ of a $2n$-gon $A_1A_2A_3\ldots A_{2n}$ inscribed in S. Show that

$$p_1p_3p_5\cdots p_{2n-1}=p_2p_4p_6\cdots p_{2n}.$$

(b) Let $p_1, p_2, p_3, \ldots, p_n$ be the distances from a point M on a circle S to the sides of an n-gon $A_1A_2A_3\ldots A_n$ inscribed in S. Let $P_1, P_2, P_3, \ldots, P_n$ be the distances from M to the sides of the circumscribed n-gon formed by the tangents to S at the points $A_1, A_2, A_3, \ldots, A_n$. Show that

$$p_1p_2p_3\cdots p_n=P_1P_2P_3\cdots P_n.$$

44. Let $a_1, a_2, \ldots, a_{n-1}, a_0$ be the lengths of the sides $A_1A_2, A_2A_3, A_3A_4, \ldots, A_nA_1$ of an n-gon $A_1A_2A_3\ldots A_n$ inscribed in a circle S and let $p_1, p_2, \ldots, p_{n-1}, p_0$ be the distances from a point M on the arc A_1A_n to the corresponding sides. Show that

$$\frac{a_0}{p_0}=\frac{a_1}{p_1}+\frac{a_2}{p_2}+\cdots+\frac{a_{n-1}}{p_{n-1}}.$$

[In particular, if the n-gon is regular, then $\frac{1}{p_0}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_{n-1}}$.]

45. Let $A_0, A_1, A_2, \ldots, A_n$ be $n+1$ points on a circle Σ of radius R.

(a) Show that if n is even, then one can always construct circles $S_0, S_1, S_2, \ldots, S_n$ tangent to Σ at $A_0, A_1, A_2, \ldots, A_n$, and such that S_1 is tangent to S_0 and S_2, S_2 is tangent to S_1 and $S_3, \ldots$, and S_0 is tangent to S_n and S_1 (Figure 60). Express the radius of S_0 in terms of R and the distances between the points $A_0, A_1, A_2, \ldots, A_n$.

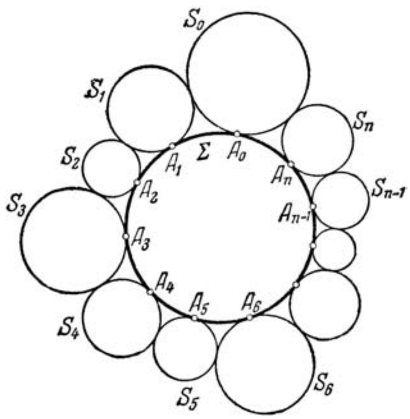


FIGURE **60**

(b) Show that if n is odd and there exist circles $S_0, S_1, S_2, \ldots, S_n$ tangent to Σ at points $A_0, A_1, A_2, \ldots, A_n$, and such that S_1 is tangent to S_0 and S_2, S_2 is tangent to S_1 and $S_3, \ldots$, and S_0 is tangent to S_n and S_1, then

$$A_0A_1 \cdot A_2A_3 \cdot A_4A_5 \cdots A_{n-1}A_n = A_1A_2 \cdot A_3A_4 \cdots A_{n-2}A_{n-1} \cdot A_nA_0.$$

The expression for the distance between the points A', B' obtained by an inversion from points A, B implies an interesting conclusion. Call the positive number

$$\frac{AC}{BC} : \frac{AD}{BD}$$

the cross ratio of four points A, B, C, D in the plane (compare this with the definition of the cross ratio of four collinear points given on p. 35 of NML 24). We show that an inversion has the following property:

D. *An inversion preserves the cross ratio of four points in the plane.*

Let A', B', C', D' be the images of A, B, C, D under an inversion. Using the formula $(*)$ we obtain

$$A'C' = AC \cdot \frac{k}{OA \cdot OC}, \qquad B'C' = BC \cdot \frac{k}{OB \cdot OC},$$

$$A'D' = AD \cdot \frac{k}{OA \cdot OD}, \qquad B'D' = BD \cdot \frac{k}{OB \cdot OD}.$$

Hence

$$\frac{A'C'}{B'C'} = \frac{AC}{BC} \cdot \frac{OB}{OA}, \qquad \frac{A'D'}{B'D'} = \frac{AD}{BD} \cdot \frac{OB}{OA},$$

whence

$$\frac{A'C'}{B'C'} : \frac{A'D'}{B'D'} = \frac{AC}{BC} : \frac{AD}{BD},$$

which was to be proved (compare this with the derivation of property C of a central projection; see p. 35 in NML 24).

We note that we cannot speak of property D of an inversion if one of the four points coincides with the center of inversion (the reason for this is that none of the points in the plane is the inversive image of the center of inversion; see p. 29).

46. Find the locus of points such that the ratio of their distances from two given points is constant.

47. Prove Ptolemy's theorem: if a quadrilateral can be inscribed in a circle, then the sum of the products of its opposite sides is equal to the product of its diagonals.

See also problem 58 below as well as problem 62(c) in NML 21.[1]

48. Use property D of inversion to solve problem 20 in Section 2.

We will now prove a property of inversion which can be viewed as a generalization of property D. Define the cross ratio of four circles S_1, S_2, S_3, S_4 as the ratio

$$\frac{t_{13}}{t_{23}} : \frac{t_{14}}{t_{24}},$$

with t_{13} defined as the length of the segment of a common tangent of S_1 and S_3 whose endpoints are the points of tangency, and with analogous definitions of t_{23}, t_{14}, and t_{24}.[2] Here some (or all) of the circles S_1, S_2, S_3, S_4 can be points ("circles of zero radius"). If, say, S_1 and S_3 are points, then t_{13} is the distance S_1S_3 between them; if S_1 is a point and S_3 is a circle,

then t_{13} is the length of the tangent from the point S_1 to the circle S_3. The following property of inversion holds:

$\overline{\mathbf{D}}$. *An inversion preserves the cross ratio of four circles provided that its center is either in the exterior or in the interior of all of them.*[3]

[If in the cross ratio of the initial circles there enters, say, the common outer tangent of S_1 and S_3, then in the cross ratio of the transformed circles there must also enter the common outer tangent of the transformed circles S_1' and S_3', and so on.[4]]

Property $\overline{\mathrm{D}}$ remains valid if some (or all) of the circles are replaced by points ("circles of zero radius"). In particular, if all circles are replaced by points, then $\overline{\mathrm{D}}$ reduces to D (and so property D can be viewed as a special case of property $\overline{\mathrm{D}}$).

We begin the proof by determining *the effect of an inversion on the length of the segment between the points of tangency of a common tangent to two circles*. Let S_1 and S_2 be two circles (neither of which passes through the center of inversion) and let S_1' and S_2' be their images under the inversion (Figure 61). Let O be the center of the inversion and k its power. Let O_1 and O_2 be the centers of S_1 and S_2 and let their radii be r_1 and r_2. Let O_1' and O_2' be the centers of S_1' and S_2' and let their radii be r_1' and r_2'. We assume that the center of inversion O lies outside S_1 and S_2 or inside S_1 and S_2.

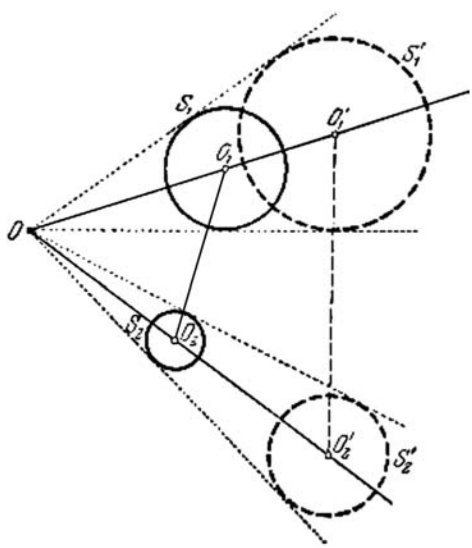
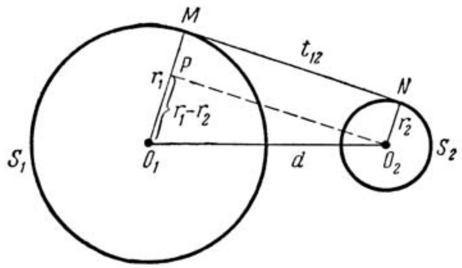


FIGURE 61



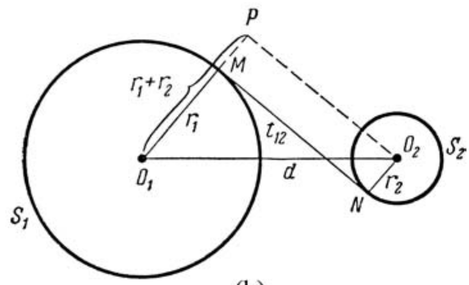


FIGURE 62

Let t_{12} be the length of the segment between the points of tangency of the common tangent to S_1 and S_2. Let t'_{12} be the length of the segment between the points of tangency of the common tangent to S'_1 and S'_2. Figures 62a and 62b imply that

$$t_{12}^2 = O_1O_2^2 - (r_1 \mp r_2)^2,$$

where the minus sign in $\mp$ corresponds to the case of a common outer tangent (Figure 62a) and the plus sign to the case of a common inner tangent (Figure 62b).

But

$$O_1O_2^2 = OO_1^2 + OO_2^2 - 2OO_1 \cdot OO_2 \cos \angle O_1OO_2.$$

Hence

$$t_{12}^2 = OO_1^2 + OO_2^2 - 2OO_1 \cdot OO_2 \cos \angle O_1OO_2 - r_1^2 - r_2^2 \pm 2r_1r_2,$$

or, after minor changes,

$$t_{12}^2 = (OO_1^2 - r_1^2) + (OO_2^2 - r_2^2) - 2OO_1 \cdot OO_2 \cos \angle O_1OO_2 \pm 2r_1r_2.$$

In much the same way we obtain

$$t_{12}'^2 = (OO_1'^2 - r_1'^2) + (OO_2'^2 - r_2'^2) - 2OO'_1 \cdot OO'_2 \cos \angle O'_1OO'_2 \pm 2r'_1r'_2.$$

As we saw when proving property B_4 of inversion (p. 9), S'_1 is centrally similar to S_1 with center of similarity O. The coefficient of similarity is k/k_1, where k_1 is the product of the distances from O to the two points of intersection of S_1 with any secant through O. k_1 is called the power of O with respect to S_1 (see the definition on p. 50). Passing a secant through the center O_1 of S_1 we find that

$$k_1 = OO_1^2 - r_1^2.$$

Thus

$$OO_1'^2 = \left(\frac{k}{k_1}\right)^2 OO_1^2, \qquad r_1'^2 = \left(\frac{k}{k_1}\right)^2 r_1^2,$$

and, similarly,

$$OO_2'^2 = \left(\frac{k}{k_2}\right)^2 OO_2^2, \qquad r_2'^2 = \left(\frac{k}{k_2}\right)^2 r_2^2,$$

where $k_2 = OO_2^2 - r_2^2$, the power of O with respect to S_2.

Now we substitute in the formula for $t_{12}'^2$ the expressions for OO_1', OO_2', r_1', and r_2' and use the fact that O_1' and O_2' lie on the lines OO_1 and OO_2, that is,

$$\angle O_1'OO_2' = \angle O_1OO_2$$

(see Figure 61).[5]

In this way we obtain:

$$\begin{aligned}
t_{12}'^2 = {} & \frac{k^2}{k_1^2}(OO_1^2 - r_1^2) + \frac{k^2}{k_2^2}(OO_2^2 - r_2^2) \\
& - 2\frac{k}{k_1}\cdot\frac{k}{k_2}\cdot OO_1 \cdot OO_2 \, \cos\angle O_1OO_2 \pm 2\frac{k}{k_1}\cdot\frac{k}{k_2} r_1 r_2 \\
= {} & \frac{k^2}{k_1^2}\cdot k_1 + \frac{k^2}{k_2^2}\cdot k_2 - 2\frac{k}{k_1}\cdot\frac{k}{k_2}\cdot OO_1\cdot OO_2 \, \cos\angle O_1OO_2 \\
& \pm 2\frac{k}{k_1}\cdot\frac{k}{k_2} r_1 r_2 \\
= {} & \frac{k^2}{k_1} + \frac{k^2}{k_2} - 2\frac{k}{k_1}\cdot\frac{k}{k_2}\cdot OO_1\cdot OO_2 \, \cos\angle O_1OO_2 \pm 2\frac{k}{k_1}\cdot\frac{k}{k_2} r_1 r_2 \\
= {} & \frac{k^2}{k_1k_2}(k_1 + k_2 - 2OO_1\cdot OO_2 \, \cos\angle O_1OO_2 \pm 2r_1r_2).
\end{aligned}$$

Since

$$\begin{aligned}
&k_1 + k_2 - 2OO_1 \cdot OO_2 \cos\angle O_1OO_2 \pm 2r_1r_2 \\
&\quad = (OO_2^2 - r_2^2) + (OO_1^2 - r_1^2) - 2OO_1 \cdot OO_2 \cos\angle O_1OO_2 \pm 2r_1r_2 \\
&\quad = t_{12}^2,
\end{aligned}$$

we obtain

$$t'_{12} = \frac{k}{\sqrt{k_1k_2}} t_{12}. \tag{$**$}$$

Note that if we replace one of the circles S_1 and S_2 by a point (if, say, we put $r_1 = 0$), or even if we replace both of these circles by points (that is, if we put $r_1 = r_2 = 0$), the argument leading to formula $(**)$ remains valid; then by the power of O with respect to the point S_1 we mean the square of the distance OS_1 ($k_1 = OO_1^2 - r_1^2$, where O_1 coincides with S_1 and $r_1 = 0$). It follows that $(**)$ is valid when one, or both, of the circles S_1 and S_2 are replaced by points. In particular, if S_1 and S_2 are points, then $(**)$ is reduced to $(*)$ (see p. 61). Hence $(*)$ can be viewed as a special case of $(**)$.

We note that since

$$r'_1 = \frac{k}{k_1} r_1, \qquad r'_2 = \frac{k}{k_2} r_2,$$

it follows that

$$r'_1r'_2 = \frac{k^2}{k_1k_2} r_1r_2, \qquad \frac{k^2}{k_1k_2} = \frac{r'_1r'_2}{r_1r_2}.$$

Consequently, we can rewrite the expression for t'_{12} in the form

$$t'_{12} = \sqrt{\frac{r'_1r'_2}{r_1r_2}}\, t_{12},$$

or in the form

$$\frac{t'_{12}}{\sqrt{r'_1r'_2}} = \frac{t_{12}}{\sqrt{r_1r_2}}.$$

The latter equality justifies the claim that *the expression* $\frac{t_{12}}{\sqrt{r_1r_2}}$ (the length of the segment between the points of tangency of a common tangent to two circles divided by the geometric mean of the radii of the circles) *is invariant under an inversion whose center is outside or inside both circles.*

Now consider four circles S_1, S_2, S_3, and S_4 (some of which may be points). Under an inversion they go over to four circles (or points) S'_1, S'_2,

S_3', and S_4'. According to $(**)$, we have:

$$t'_{13} = \frac{k}{\sqrt{k_1 k_3}}\, t_{13},$$

$$t'_{14} = \frac{k}{\sqrt{k_1 k_4}}\, t_{14},$$

$$t'_{23} = \frac{k}{\sqrt{k_2 k_3}}\, t_{23},$$

$$t'_{24} = \frac{k}{\sqrt{k_2 k_4}}\, t_{24},$$

whence

$$\frac{t'_{13}}{t'_{23}} : \frac{t'_{14}}{t'_{24}} = \frac{t_{13}}{t_{23}} : \frac{t_{14}}{t_{24}},$$

which is what we wished to prove (compare this proof with the proof of property D of an inversion).

49. Find the locus of points M such that the ratio of the lengths of the tangents from M to two given circles is constant.

See also problem 64 in NML 21 and problem 39(b) in Section 3 of this book. If S_1 and S_2 are "circles of zero radius," that is, points, then our problem reduces to problem 46.

50. Prove the following generalization of Ptolemy's theorem: if four circles S_1, S_2, S_3, and S_4 are tangent to the same fifth circle (or line) Σ (Figure 63), then we have the relation

$$t_{12}t_{34} + t_{14}t_{23} = t_{13}t_{24},$$

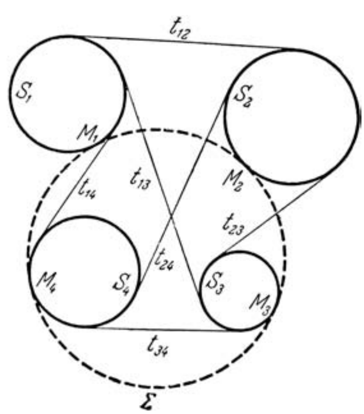


FIGURE 63

where t_{12} is the length of the common tangent to S_1 and S_2 and t_{13}, t_{14}, t_{23}, t_{24}, and t_{34} have analogous meanings.

[We are assuming that the points M_1, M_2, M_3, M_4 at which the circles S_1, S_2, S_3, S_4 touch the circle Σ are located on Σ in the order M_1, M_2, M_3, M_4. We are also assuming that, say, t_{12} is the segment of the outer tangent to S_1 and S_2 if both of these circles are in the exterior or the interior of Σ and of the inner tangent otherwise; if Σ is a line, then t_{12} is the segment of the outer tangent to S_1 and S_2 if both of these circles are in the exterior or the interior of Σ and of the inner tangent otherwise.]

The theorem in problem 50 remains valid if some, or all, of the circles S_1, S_2, S_3, S_4 are replaced by points ("circles of zero radius"). If all of these circles are replaced by points, then we end up with Ptolemy's theorem (see problem 47).

51. Show that if four circles S_1, S_2, S_3, S_4 pass through the same point of Σ, then the lengths of the common tangents to these circles satisfy the relation stated in problem 50.

If a point is viewed as a special case of a circle ("a circle of zero radius"), then the theorem in problem 51 should be viewed as a special case of the theorem in problem 50. Note, however, that while in the statement of problem 50 we "lumped together" the case when Σ is a circle and the case when it is a line ("a circle of infinite radius"), the proposition in problem 51 requires a separate proof. This is due to the fact that in the theory of inversion circles and lines have the same status (either can go over to the other under an inversion), whereas points and circles do not. That is why when we are dealing with problems involving inversion it is often possible to replace a circle by a line (that is, view a line as a special case of a circle) or a line by a circle without affecting the solution of the problem. But in cases in which a point takes the place of a circle in the statement of a problem a separate proof is required.[6]

We note that the converse of the propositions in problems 50 and 51 holds. Specifically, if the lengths of the segments of common tangents to four circles satisfy the relation in problem 50, then these four circles are tangent to a fifth circle, or to a line, or pass through a single point (see problem 62(b) in Section 5 of this book).

52. Prove that if four pairwise intersecting circles S_1, S_2, S_3, S_4 are tangent to a circle Σ or pass through the same one of its points, then the

angles between them satisfy the relation

$$\sin\frac{\alpha_{12}}{2}\sin\frac{\alpha_{34}}{2}+\sin\frac{\alpha_{14}}{2}\sin\frac{\alpha_{23}}{2}=\sin\frac{\alpha_{13}}{2}\sin\frac{\alpha_{24}}{2},$$

where, say, α_{12} is the angle between the circles S_1 and S_2 (more precisely, the angle between their radii drawn to a point of their intersection).

[Here it is assumed that the points of tangency between the circles S_1, S_2, S_3, S_4 and the circle or line Σ are located on Σ in the order specified in problem 50; if Σ is a point, then the radii $O_1\Sigma$, $O_2\Sigma$, $O_3\Sigma$, $O_4\Sigma$ are located around Σ in the order ΣO_1, ΣO_2, ΣO_3, ΣO_4.]

When we used affine transformations in geometric problems involving proofs in Sections 1–3 of NML 24 the aim—for the most part—was to choose an affine transformation that would simplify the figure involved. On the other hand, when we used polar transformations of the plane in problems in Section 4 of NML 24 the aim was different: there the transformed figure of an established theorem yielded a new theorem that required no proof—its validity was implied by the validity of the initial theorem. To restate this somewhat differently we can say that the main advantage of using polar transformations was to obtain from established theorems completely new theorems.

In this book, in problems 1–16 in Section 1 and in problems 42–52 in this section, inversions were used to simplify the figures associated with the relevant problems, and thus in a manner analogous to the manner in which affine transformations were used in Sections 1–3 of NML 24. But it is clear that inversions can also be used to obtain new theorems from established ones. After all, inversions can change figures rather radically (they can take lines to circles), which means that the new theorem obtained in this way can differ significantly from the original theorem. Problems 53–59 below exemplify such use of inversions.[7]

53. (a) We know that the bisectors of the angles formed by two intersecting lines are the loci of the centers of the circles that are simultaneously tangent to these two lines. If we invert the figure associated with this theorem, then we obtain a theorem associated with the inverted figure. State this theorem.

(b) The loci of the centers of the circles that intersect two intersecting lines l_1 and l_2 are also the bisectors of the angles formed by l_1 and l_2.

State the theorem associated with the figure obtained by inverting the figure associated with the original theorem.

(c) Construct a circle that cuts four given pairwise intersecting circles at the same angle.

54. State the theorem obtained by applying an inversion to the theorem which states that the sum of the angles of a triangle equals 180°.

55. What theorem does one obtain by applying an inversion to the theorem:

(a) The altitudes of a triangle are concurrent.

(b) The angle bisectors of a triangle are concurrent.

56. Into what theorem does the theorem on the Simson line (problem 61 in NML 21) go over under an inversion wih center at P?

57. Into what theorem does the following theorem go over under an inversion: A circle is the locus of points equidistant from a point (definition of a circle).

58. (a) Into what theorem does the following theorem go over under an inversion: Each side of a triangle is less than the sum of its two other sides.

(b) Prove the converse of Ptolemy's theorem (problem 47): If the sum of the products of the opposite sides of a quadrilateral is equal to the product of its diagonals, then we can circumscribe a circle about it.

59. Into what theorems do the following theorems go over under an inversion:

(a) the theorem of Pythagoras;

(b) the law of cosines;

(c) the law of sines.

Property B of inversion asserts that this transformation takes circles to circles (here the term "circle" includes lines, viewed as "circles of infinite radius"). All transformations with these properties are called *circular transformations*. The simplest examples of circular transformations are isometries and similarities. Isometries and similarities take lines to lines, and thus are circular transformations as well as affinities (see p. 18 in NML 24). Inversions are also circular transformations, but they are more complicated than either isometries or similarities (of course, inversions are not affinities).

One can show that a transformation that is both affine and circular is a similarity, and that a circular transformation that is not an affinity can be reduced to an inversion. More precisely, the following two remarkable theorems hold:

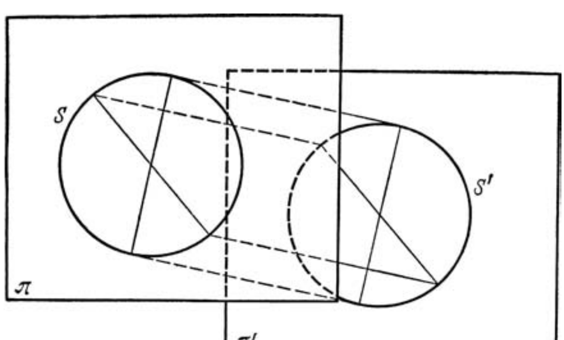
Theorem 1. *Every circular transformation of the plane which is also an affinity is a similarity.*

Theorem 2. *Every circular transformation of the plane which is not a similarity can be realized by an inversion followed by a similarity.*

Theorem 2 implies that we can define a circular transformation as an inversion possibly followed by a similarity (in this connection see pp. 68–70 in NML 8). In particular, it implies that circular transformations have the properties A, B, C, D of inversions (see pp. 6–11 and p. 63), for, obviously, similarities also have these properties. Finally, Theorem 2 enables us to describe the nature of a product of two (or more) inversions: such a product is an inversion followed by a similarity. This is so because such a product is a circular transformation of the plane.

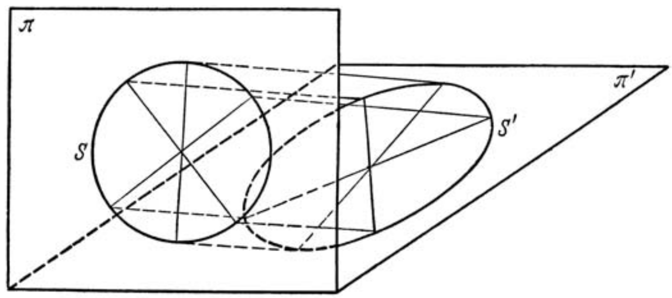
Proof of Theorem 1. We know that every affinity of the plane is a parallel projection of the plane to itself followed, possibly, by a similarity (see Theorem 2 on p. 18 of NML 24). Thus all we need to clarify is when a parallel projection of the plane to itself (defined on p. 18 of NML 24) takes circles to circles.

It is obvious that under a parallel projection of a plane π to a parallel plane π' a circle in π goes over to a circle in π'; in fact, such a projection is just a translation of π in space in the direction of the projection to its coincidence with π' (Figure 64a).[8] On the other hand, if π is not parallel to π', then a parallel projection cannot take a circle to a circle. In fact, under such a projection, the diameter of the circle parallel



(a)

FIGURE 64



(b)

FIGURE 64

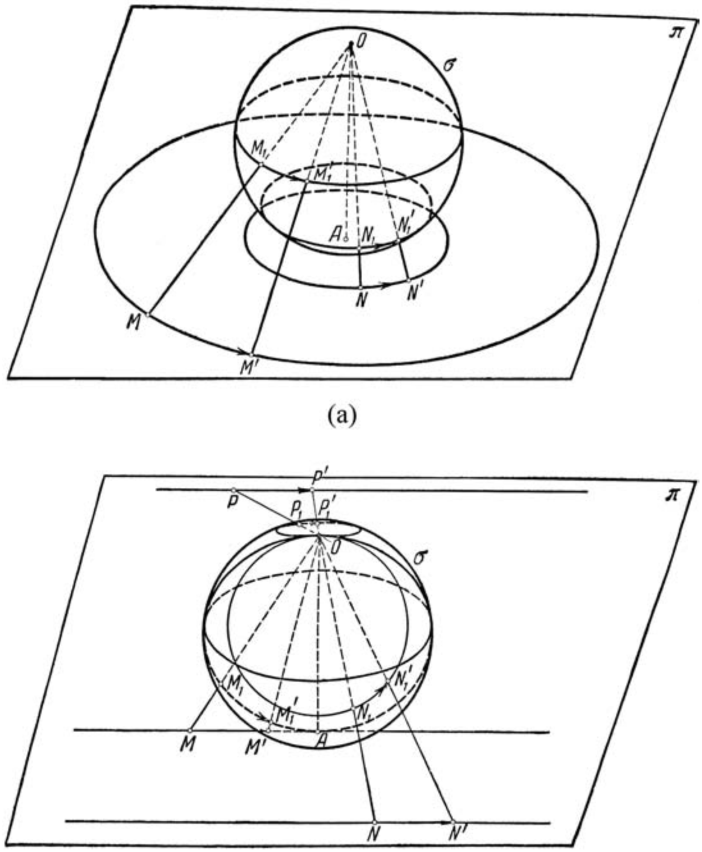
to the line of intersection of π and π' goes over to a segment of the same length, whereas the other diameters go over to segments of different lengths. This means that the curve which is the image of the circle is not a circle (Figure 64b). Thus *if a parallel projection of a plane to itself is to take circles to circles, it is necessary that its new position, from which the parallel projection to the initial position is carried out, be parallel to that initial position*. Since in that case parallel projection can be replaced by a translation in space, it follows that such a parallel projection of a plane to itself is equivalent to an isometry. This fact, and Theorem 2 on p. 18 in NML 24, imply Theorem 1.

Proof of Theorem 2. (This proof is much more difficult than the proof of Theorem 1.) What plays a key role in this proof is the stereographic projection of a sphere to a plane (see p. 55 in NML 24); more specifically, the fact that *a stereographic projection maps a circle* (throughout this section the term "circle" includes lines) *in the plane to a circle on the sphere* (see Theorem 2 and its proof on pp. 55–57 in NML 24).

Let $\mathbb{K}$ be a transformation of the plane π. We use the stereographic projection to map π on the sphere σ. Then to the transformation $\mathbb{K}$ of π there corresponds a transformation $\overline{\mathbb{K}}$ of σ. For example, to a rotation of π about the point A at which the sphere σ touches π there corresponds a rotation of σ about its diameter through A (Figure 65a), and to a translation of π in a direction a there corresponds the transformation of σ in which each point of the sphere moves on a circle passing through the center of projection and located in a plane parallel to the direction a (Figure 65b). The fundamental property of a stereographic projection implies that *if a transformation* $\mathbb{K}$ *of the plane is circular, then so too is the transformation* $\overline{\mathbb{K}}$ *of the sphere*. Conversely, if a transformation $\overline{\mathbb{K}}$ of the sphere is circular, then so too is the transformation $\mathbb{K}$ of the plane.

Now let $\mathbb{K}$ be a circular transformation of the plane π and $\overline{\mathbb{K}}$ the corresponding transformation of the sphere σ. *If* $\overline{\mathbb{K}}$ *fixes the center of projection O then* $\mathbb{K}$ *is a similarity*.

In fact, we saw in Section 3 of NML 24 that a stereographic projection takes the lines in π to circles of σ passing through O (see Figure 53 on p. 55 of NML 24).



(a)

(b)

FIGURE 65

Clearly, the transformation $\overline{\mathbb{K}}$ of σ that fixes O takes each circle through O to a circle through O. This implies that the transformation $\mathbb{K}$ of π takes every line in π to a line in π, that is, $\mathbb{K}$ is both circular and affine. But this is possible only if $\mathbb{K}$ is a similarity (see Theorem 1).

Assume that a circular transformation $\overline{\mathbb{K}}$ of σ does not fix O. Let O_1 be the image of O under $\overline{\mathbb{K}}$ and let δ be the diametric plane of σ perpendicular to the segment OO_1 (Figure 66). Let $\overline{\mathbb{I}}$ be the reflection of σ in δ (that is, $\overline{\mathbb{I}}$ is the transformation of σ that takes a point $\overline{M}$ to the point $\overline{M'}$ symmetric to M with respect to δ). $\overline{\mathbb{I}}$ is a circular transformation of the sphere that also takes O_1 to O. Let $\overline{\mathbb{K}}_1$ be a transformation of the sphere such that $\overline{\mathbb{K}}$ is the product of $\overline{\mathbb{I}}$ and $\overline{\mathbb{K}}_1$ (that is, $\overline{\mathbb{K}}$ is the result of the successive application of $\overline{\mathbb{I}}$ and $\overline{\mathbb{K}}_1$).[9] It is not difficult to see that $\overline{\mathbb{K}}_1$ is also a circular transformation of the sphere: if $\overline{\mathbb{I}}$ takes a circle $\overline{S}$ to a circle $\overline{S}_1$ and

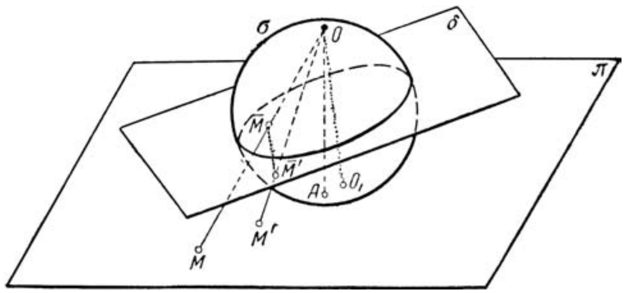


FIGURE 66

$\overline{\mathbb{K}}$ takes $\overline{S}$ to $\overline{S'}$, then $\overline{\mathbb{K}}_1$ takes $\overline{S}_1$ to $\overline{S'}$. Also, $\overline{\mathbb{K}}_1$ fixes O (for both $\overline{\mathbb{K}}$ and $\overline{\mathbb{I}}$ take O_1 to O).

Since $\overline{\mathbb{K}}$ can be written as the product $\overline{\mathbb{K}}_1\overline{\mathbb{I}}$, it follows that $\mathbb{K}$ can be written as the product $\mathbb{K}_1\mathbb{I}$, where $\mathbb{I}$ corresponds to $\overline{\mathbb{I}}$ and $\mathbb{K}_1$ to $\overline{\mathbb{K}}_1$ under the stereographic projection. Since $\overline{\mathbb{K}}_1$ fixes O, $\mathbb{K}_1$ is a similarity (see above). We will now prove that $\mathbb{I}$ is an inversion. This will complete the proof of Theorem 2.

Let $\overline{S}$ be a circle on the sphere perpendicular to the circle $\overline{\Sigma}$ (Figure 67). This means that the plane of $\overline{S}$ is perpendicular to the plane δ. Let S be the circle (or line) that is the image of $\overline{S}$ under the stereographic projection. Clearly, the reflection $\overline{\mathbb{I}}$ takes $\overline{S}$ to itself. Hence $\mathbb{I}$ takes S to itself. We will show that S is perpendicular to Σ.

Let $\overline{B}\,\overline{T}$ and $\overline{B}\,\overline{T_1}$ be tangent to $\overline{\Sigma}$ and $\overline{S}$ at their point of intersection $\overline{B}$; $\overline{B}\,\overline{T_1} \perp \overline{B}\,\overline{T}$. The central projection with center O maps the lines $\overline{B}\,\overline{T}$ and $\overline{B}\,\overline{T_1}$ on the tangents BT and BT_1 to the circles Σ and S at their point of intersection B. Let P and Q denote the points of intersection of $\overline{B}\,\overline{T}$ and $\overline{B}\,\overline{T_1}$ with the plane π, and let Q_1 denote the point of intersection of $\overline{B}\,\overline{T_1}$ with the plane π_1 parallel to π and passing through O (the plane π_1 does not appear in Figure 67). Since the triangles $\overline{B}OQ_1$ and $\overline{B}BQ$ are similar (for $Q_1O \parallel QB$ as the lines of intersection of the plane $\overline{B}QB$ with the two parallel planes π_1 and π) and $Q_1\overline{B} = Q_1O$ (they are two

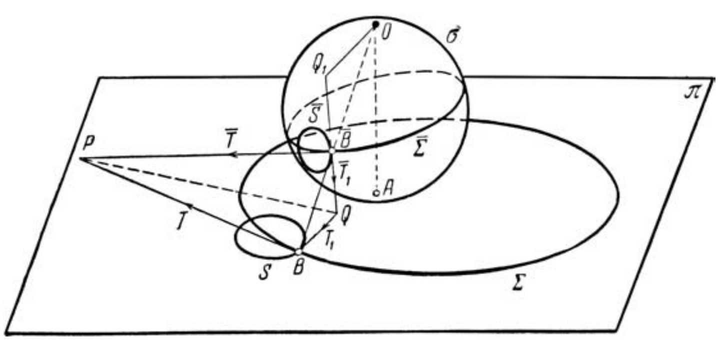


FIGURE 67

tangents to the sphere σ issuing from Q_1), it follows that $QB = Q\overline{B}$; we show in an analogous way that $PB = P\overline{B}$. Now the congruence of the triangles PBQ and $P\overline{B}Q$ implies that[10]

$$\angle PBQ = \angle P\overline{B}Q = 90^\circ,$$

that is, that the circles S and Σ are perpendicular to one another.

We see that $\mathbb{I}$ fixes all points of the circle Σ and that it takes every circle S perpendicular to Σ to itself.[11] Hence $\mathbb{I}$ takes all circles perpendicular to Σ and passing through the fixed point A to themselves. Keeping in mind that $\mathbb{I}$ interchanges the exterior and interior of Σ, we are bound to conclude that $\mathbb{I}$ takes A to the point A' symmetric to A with respect to Σ, that is, that $\mathbb{I}$ is a reflection in Σ.

We note that the presence in the plane of a singular point with no image in the plane under reflection in a circle does not, strictly speaking, allow us to regard an inversion as a transformation of the plane in the usual sense of that term (a transformation is usually thought of as a one-to-one onto mapping). This being so, our formulation of the fundamental Theorem 2 is not quite accurate. It is easy to see that a transformation of the plane, as usually defined, cannot take a circle, a closed curve, to a line, which is not a closed curve. Hence all transformations of the plane—in the usual sense of the term—which take circles and lines to circles and lines, and are therefore circular transformations (see Theorem 1), must take lines to lines and circles to circles, that is, must be similarities. The transformation we called circular can be rigorously defined—in a manner analogous to projectivities—as transformations that take a region Ω in the plane to a region Ω' in the plane and a circle or line intersecting Ω to a circle or line intersecting Ω' (a generalized circular transformation [12]). We will not consider this issue in detail.

Notes to Section 4

[1] Another proof of Ptolemy's theorem, and a number of its applications, are found in the book by D. O. Shklyarskiĭ et al., *Collected Problems and Theorems*, Part 2 (Russian).

[2] If we take t_{13} to be the length of the segment of the common outer tangent of S_1 and S_3, or the length of the segment of their common inner tangent, and introduce analogous definitions of t_{23}, t_{14}, and t_{24}, then we obtain a total of 16 cross ratios of the four given pairwise disjoint circles.

[3] If the center of inversion O is a point of one of the circles under consideration, then its image under an inversion is a line, and property $\overline{D}$

is without meaning. If O lies outside some of the four circles and inside others, then property $\overline{D}$ does not hold either. For example, if the center of inversion O is outside S_1 and inside S_3 and these two circles are exterior to one another, then an inversion maps them on circles S_1' and S_3' of which one is interior to the other. But then S_1' and S_3' have no common tangent and it is therefore impossible to define their cross ratio.

[4] If all four circles are exterior to one another, then an inversion with center in the exterior of all of them preserves all their 16 cross ratios (see Note 2).

[5] Here we use the fact that both similarity coefficients, $\frac{k}{k_1}$ and $\frac{k}{k_2}$, have the same sign. This is so because, by assumption, O lies outside S_1 and S_2 or inside S_1 and S_2.

[6] Exceptions are the formula $(**)$ and property $\overline{D}$ of inversion which, as shown earlier, retain their validity if some, or all, circles are replaced by points. This allows us to replace some of the circles by points in problems based on $(**)$ or on $\overline{D}$. Specifically, that is why we can assume in the statement of problem 50 that some of the circles S_1, S_2, S_3, and S_4 are "circles of zero radius," that is, points. On the other hand, we must not replace any of the circles S_1, S_2, S_3, and S_4 by lines, for then $(**)$ and $\overline{D}$ lose meaning.

[7] Note that, strictly speaking, in all such problems we must apply inversion twice (see the Note on p. 81 of NML 24).

[8] See the Note on p. 10 of NML 24.

[9] In other words, if $\overline{\mathbb{I}}$ takes a point $\overline{M}$ on σ to $\overline{M}_1$ and $\overline{\mathbb{K}}$ takes $\overline{M}$ to $\overline{M}'$, then $\overline{\mathbb{K}}_1$ takes $\overline{M}_1$ to $\overline{M}'$.

[10] It is clear that the proof of the equality of the angles $\overline{TBT_1}$ and TBT_1 holds when $\overline{TBT_1}$ is not a right angle. This implies that *stereographic projection preserves angles* (that is, the angle between circles $\overline{\Sigma}$ and $\overline{S}$ on σ is equal to the angle between their images Σ and S (which may be circles or lines) in π). [Quite generally, stereographic projection of σ on π is a conformal mapping (see Note 7 in Section 1).]

[11] It is easy to see that every circle in a plane perpendicular to Σ can be obtained by stereographic projection of a circle $\overline{S}$ perpendicular to $\overline{\Sigma}$ (this follows, in particular, from the fact that in this way we can obtain a circle perpendicular to Σ and intersecting it in any preassigned pair of points); it follows that $\mathbb{I}$ takes every circle perpendicular to Σ to itself.

[12] Rigor requires that we speak of circular transformations of a conformal plane (see Note 12 in Section 1). This was our implicit approach in all

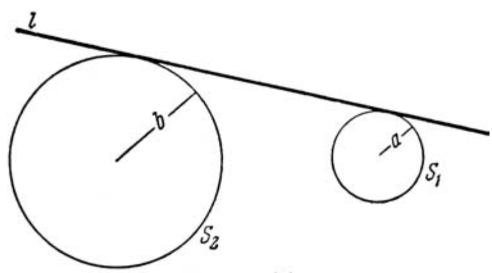
preceding arguments. (We note that a stereographic projection takes the points of the sphere σ to points of the conformal plane π and, in particular, the point O of the sphere to the "point at infinity" of the plane π.)

5
Axial circular transformations

A. Dilatation

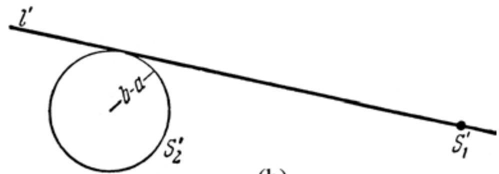
At the beginning of this section we consider a simple transformation which sometimes helps solve geometric problems. The properties of this transformation are significantly different from the properties of the transformations we have studied thus far.

We recall the following well-known problem whose solution is given in Kiselev's school textbook of geometry: *Construct a common tangent to two given circles*. The idea of the solution is the following. Suppose the problem has been solved (Figure 68a; for definiteness we limit ourselves here to the case of a common outer tangent). Now we transform our figure by decreasing the radii a and b ($a < b$) of the given circles S_1 and S_2 by a without changing their centers. Then S_1 goes over to the point S_1' and S_2 to the circle S_2' with radius $b - a$ (Figure 68b). It is clear that the



(a)

FIGURE 68



(b)

FIGURE 68

tangent l' from S_1' to S_2' is parallel to the common tangent l of S_1 and S_2 and the distance between l and l' is a. Using the well-known construction of a tangent from a given point to a given circle we construct l'. Then we translate l' by a in the direction perpendicular to l' and in this way obtain the required common tangent l.

The transformation that takes Figure 68b to Figure 68a is called a *dilatation* by a; it increases the radii of all circles by a (so that points—circles of zero radius—go over to circles of radius a) and translates all lines by a in a direction perpendicular to them. The transformation that takes Figure 68a to Figure 68b is called a dilatation by the negative quantity $-a$, or a compression; it decreases the radii of all circles by a.

Before going on to study the properties of dilatations we must consider a fact that is essential for the rest of this section. This fact is that our definition calls for additional clarifications without which it is meaningless. Our definition does not explain what are the images of circles with radii smaller than a (for example, points) under a negative dilatation, that is, under a compression, by $-a$. That is not all. Even if we limited ourselves to positive dilatations we would still be in trouble. Specifically, we said that a dilatation by a translates a line l in a direction perpendicular to l through a distance a. But there are two such lines (Figure 69), so the question to be answered is which of these two lines is to be chosen as the image of l under the dilatation.

In the problem analyzed earlier (Figure 68) what helped us make the choice was the fact that l was tangent to S_2; we translated l so that its "dilated" version (here, actually, its "compressed" version) should be

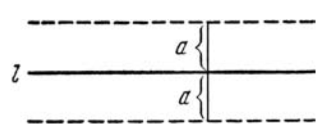


FIGURE 69

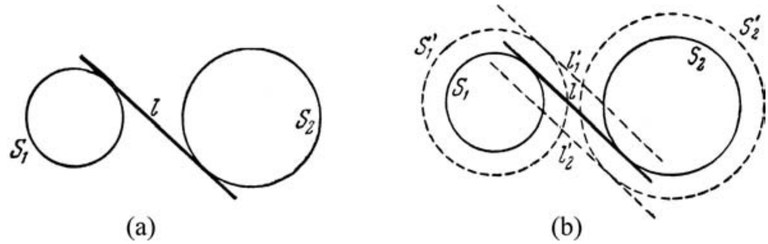


FIGURE 70

tangent to S_2'. However, it is clear that this approach does not settle the matter in the general case. If we consider Figure 70a, in which, to begin with, the line l is tangent to two circles, we would have to conclude that under a dilatation it "splits in two," by going over to two different lines l_1' and l_2' (Figure 70b). An additional difficulty is that we don't know how to determine the image under a dilatation of a line not tangent to either of the circles in the figure. Thus to obtain a definition of a dilatation applicable to all situations we must supplement our earlier definition by specifying, in addition to the "size" a, in which of the two possible directions an arbitrary line is moved.

Given a line, we cannot distinguish one of its two sides from the other. To make such a distinction we must first assign a direction to the line itself; it is only then that we can speak of moving that line "to the right" or "to the left." (We cannot speak of the "right side" or "left side" of a street without first indicating the direction of motion of a person walking on that street.) We will call a line with an assigned direction a *directed line*, or an *axis*; we will usually use the first of these two terms. Thus a complete definition of a dilatation requires the introduction of the notion of a directed line. If we assume that all lines are directed, we can state that *under a dilatation by a positive quantity a every line is moved by a in a direction perpendicular to itself in a definite one of the two possible directions* (in the sequel we will choose the *right* direction); now each directed line will go over to a definite (directed) line and no two (directed) lines will go over to the same (directed) line. Clearly, two lines l_1 and l_2 that differ only by their directions will go over under a dilatation to two different lines l_1' and l_2' (Figure 71a); now we can explain the "doubling" of a line l under a dilatation (Figure 70a). Conversely, the two different lines l_1 and l_2 in Figure 71b go over under the relevant dilatation to two lines l_1' and l_2' that differ only by their directions and are otherwise coincident.

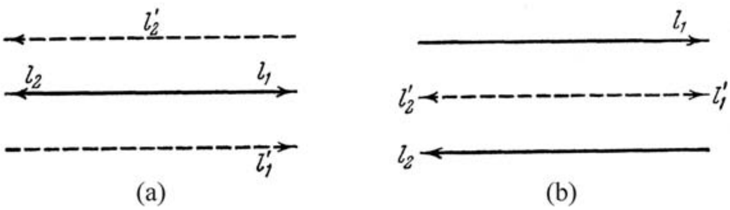


FIGURE 71

We give another justification for the introduction of the notion of a directed line. Suppose that a dilatation moves a horizontal line upward by a. It is natural to require that lines neighboring the horizontal line (that is, intersecting it at a small angle) should also move in a direction close to the upward direction and not in the opposite direction (what enters here is the notion of continuity of a transformation). In much the same way we can now define the direction of motion of lines close to the lines which are close to the horizontal line, and so on. We consider the pencil of lines obtained from the horizontal line by rotating it counterclockwise through all angles between 0° and 180° (Figure 72). If we require that neighboring lines move in neighboring directions, then the fact that the horizontal line moves upward determines uniquely the direction of motion of all lines of the pencil. Specifically, lines neighboring the horizontal move upward and slightly to the left (since the lines are not directed, "to the left" does not mean "to the left of a line" but "to the left as interpreted by a person looking at the plane"). The line making a 45° angle with the horizontal line moves upward and to the left, the vertical—somewhat to the left, and the line that forms a 180° angle with the horizontal line (that is, the horizontal line itself)—somewhat down. To sum up: if a dilatation is to be a continuous transformation (discontinuous transformations, marked by the fact that the images of neighboring lines have radically different orientations, are of no interest in geometry), then, under a dilatation, the horizontal line must move up as well as down, as required by the transformation depicted in Figure 72. The only

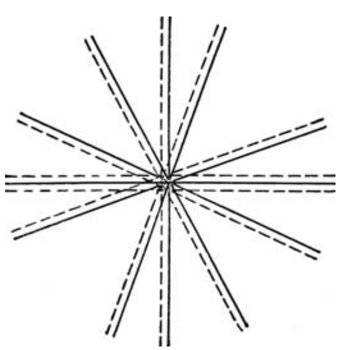


FIGURE 72

way out of this difficulty is to think of lines as "double," that is, as distinguished not only by their positions in the plane but also by their directions. If we introduce this convention, then a directed horizontal line will, upon rotation through 180°, go over to a line with opposite direction. When subjected to a dilatation, the latter line will move downward, as required.

Throughout this section, the term "line" will always denote a directed line (an "axis"). [The only exceptions are the statements of problems; here lines are not directed.] We will say about two directed lines that they are parallel only if they are parallel in the usual sense and have the same direction (Figure 73a); thus the lines in Figure 73b are not regarded as parallel. By the *angle* between two directed lines we mean the angle between the directed

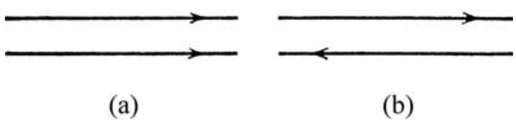


FIGURE 73

rays determined by them (Figure 74a). On the other hand, we can choose as the angle between two nondirected lines either one of the two angles formed by them ($\angle AOC$ or $\angle AOB$ in Figure 74b).

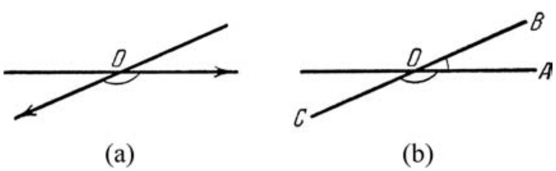


FIGURE 74

The introduction of directed lines eliminates the first of the two difficulties associated with the definition of a dilatation given at the beginning of this section. Now we can reformulate this definition as follows: consider a figure T consisting of a finite number of circles and directed lines (axes). *By a dilatation of T by a positive quantity a we mean a transformation of T in which each circle is replaced by a circle with the same center and radius increased by a* (in particular, each point—circle of zero radius—is replaced by a circle of radius a and center at that point) *and each* (directed) *line moves to the right by a*. We cannot as yet define a dilatation by a negative quantity $-a$. This is so because it is not clear how to decrease

the radius r of a circle if $r < a$. Formally, we could say that a dilatation by a negative quantity $-a$ takes a circle with radius $r < a$ to a circle with radius $r - a = -(a - r)$, but thus far it is not clear what this is to mean.

A way out of this difficulty is suggested by the way in which we eliminated the difficulty associated with the dilatation of lines. We will assume that all circles in the plane are double circles, that is, that to a given center and radius there correspond two oppositely directed circles. Just as in the case of lines, the direction of a circle in a figure will be indicated by an arrow. We will speak of directed circles, or cycles. One is free to adopt either direction—clockwise or counterclockwise—as positive. We choose the counterclockwise direction as positive and the clockwise direction as negative. We will say that the radius of a positively directed circle is positive and the radius of a negatively directed circle is negative.

Now we can give a final definition of a dilatation. *By a dilatation by a* (which can be positive or negative) *we mean a transformation of a figure* consisting of directed lines (axes) and directed circles (cycles) *in which every directed circle with center O and radius r goes over to the directed circle with the same center O and radius $r + a$ and every directed line moves to the right by a*. We add that by a translation by a negative distance $-a$ we mean a translation by a in the opposite direction, that is, to the left. Clearly, if a dilatation by a takes a figure T to a new figure T', then the dilatation by $-a$ takes T' to T. It is easy to see that a dilatation by a takes two circles S_1 and S_2 that differ only by their directions—that is, their radii are r and $-r$—to two different circles S_1' and S_2' with radii $r + a$ and $-r + a$ (the circle has "split in two"; Figure 75a depicts the case $a < r$ and Figure 75b depicts the case $a > r$); conversely, the dilatation by $-a$ will take the different circles S_1' and S_2' to circles S_1 and S_2 that differ only by direction.

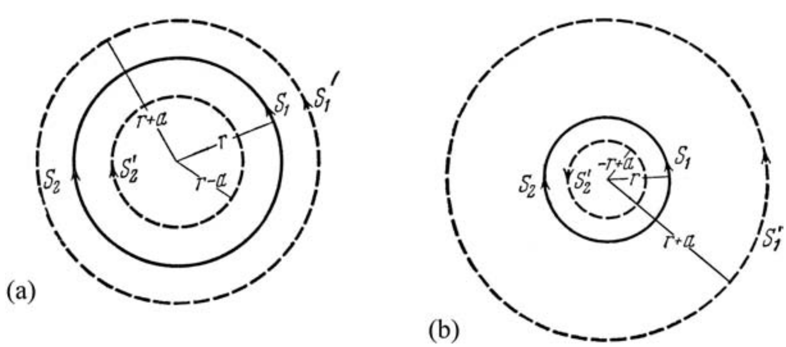


FIGURE 75

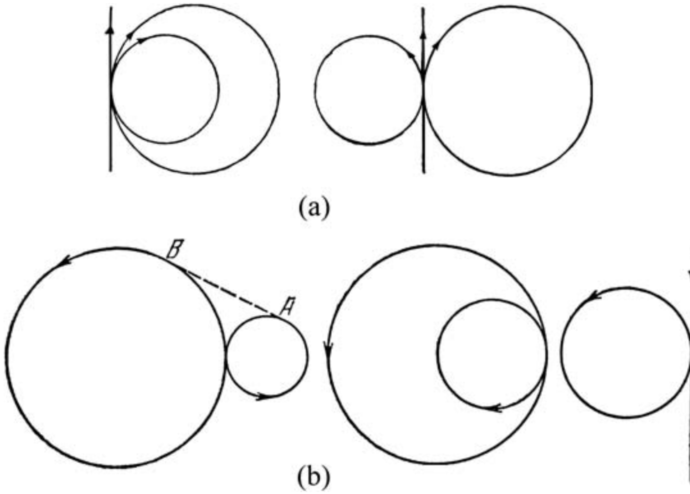


FIGURE 76

Throughout this section the term "circle" will always denote a directed circle (a "cycle"). [The only exceptions are the statements of problems; here circles are not directed.] Of two (directed) circles, or of a (directed) circle and a (directed) line, we will say that they are tangent only if their directions at the point of tangency coincide (Figure 76a); thus the two circles in Figure 76b, and the circle and line in that figure, are not tangent. Under these conditions two (directed) circles cannot have more than two common tangents; the common tangents are outer if the circles have the same directions (Figure 77a) and inner otherwise (Figure 77b). In this section we will regard points as "cycles of zero radius." Obviously, points have no direction. We will say that a circle (or line) "touches" a point A if it passes through it.

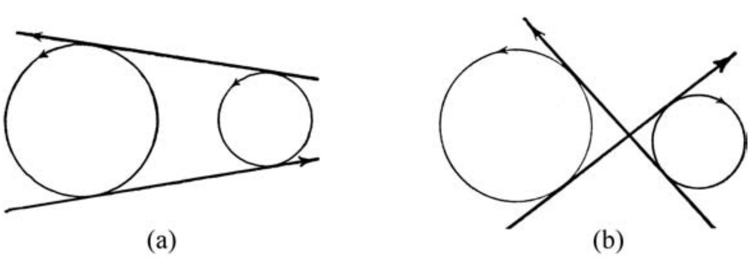
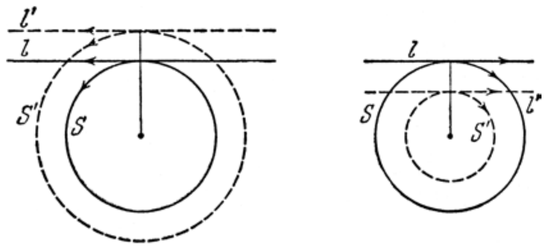


FIGURE 77



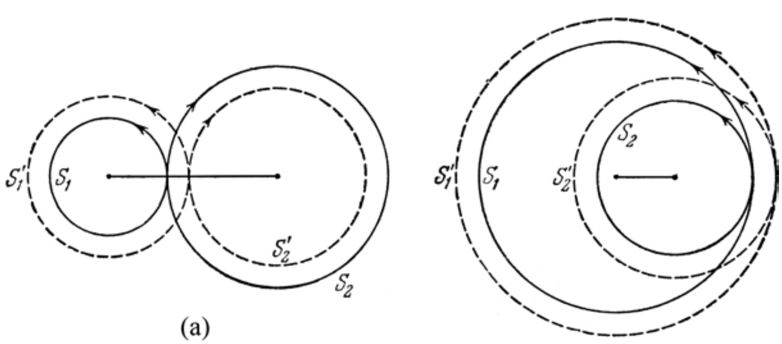
(a) (b)

FIGURE 78

Now we state and prove two properties of a dilatation, labelled A and B, and then, after introducing the notion of tangent distance, we state and prove yet another of its properties, labelled C.

A. *A dilatation takes a tangent circle and line to a tangent circle and line* (Figures 78a and 78b). Figures 78a and 78b make this property rather obvious. Define the distance from a point to a directed line as positive if the point is to the left of the line and as negative otherwise. *A* (directed) *circle is tangent to a* (directed) *line if the distance from the center of the circle to the line is equal to the* (positive or negative) *radius of the circle* (Figures 78a and 78b). Under a dilatation by a the distance from the center of the circle S to the line l, as well as the radius of S, are increased by a. This proves that a dilatation preserves the tangency of a line and a circle.

B. *A dilatation takes tangent circles to tangent circles* (Figures 79a and 79b). Figures 79a and 79b make this property rather obvious. It is easy



(a) (b)

FIGURE 79

to verify that *two* (directed) *circles are tangent if and only if the distance between their centers is equal to the difference of their radii* (Figures 79a and 79b). Now a dilatation by a increases the radii of all circles by a, while the distances between their centers remain unchanged. This proves that a dilatation preserves the condition for the tangency of two circles.

To better understand the above, we consider once more the problem of constructing a common tangent to two circles S_1 and S_2 with radii a and b. We assume that the circles are directed, so that their radii can be positive or negative. In accordance with the definition of tangents to directed circles, the common tangents are outer if the signs of a and b are the same (Figure 80a) and inner if they are different (Figure 80b). To solve the problem we need only apply a dilatation by $-a$ or by $-b$. This dilatation takes one of the circles to a point. The problem of constructing a tangent

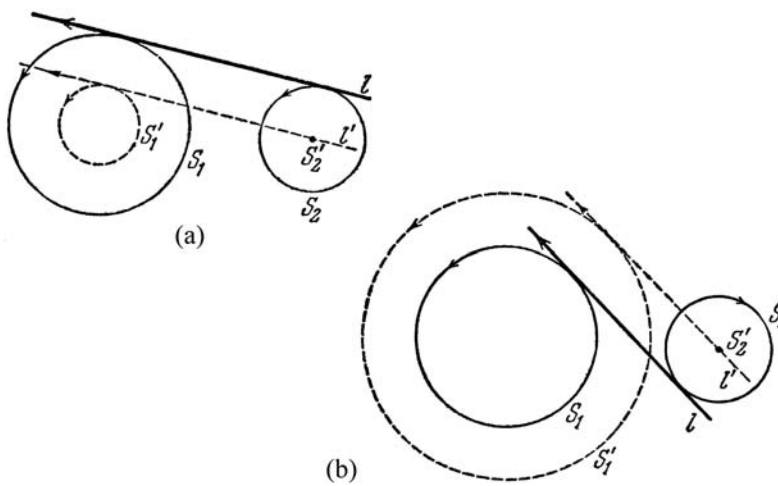
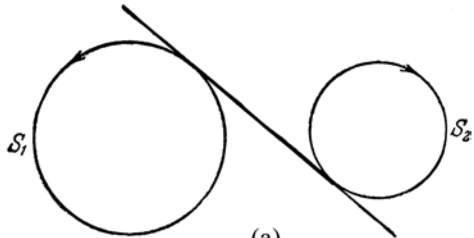


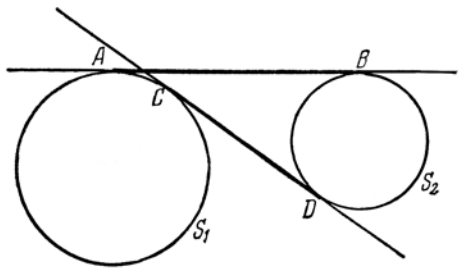
FIGURE 80

to two circles is reduced to the problem of constructing a tangent from a given point to a given circle. This example demonstrates the advantage of the introduction of directed lines and circles; the descriptions of the different cases shown in Figures 80a and 80b are the same (of course, in Kiselev's "Geometry" these two cases are considered separately).

Before discussing a third property of a dilatation we introduce the concept of *tangent distance* which will play a major role in this section. By the tangent distance between circles S_1 and S_2 we mean the length of the segment of their common tangent between the points of tangency (Figure 81a). The tangent distance between two nondirected circles can take on two different values (AB or CD in Figure 81b).



(a)



(b)

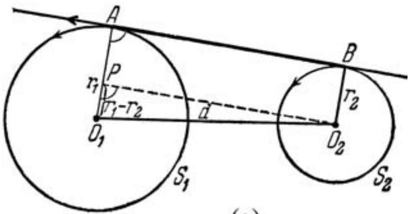
FIGURE 81

On the other hand, two directed circles cannot have more than two common tangents, and the lengths of the segments of these tangents are the same (see Figure 77). [We recall that we can assign one of two values to the angle between two nondirected lines, whereas the angle between directed lines is uniquely determined (see Figures 74a and 74b).] If two (directed) circles have no common tangents (if, say, one is inside the other), then these circles have no tangent distance (just as we cannot define an angle between nonintersecting circles). The tangent distance between two directed circles is zero if and only if they are tangent to one another (see Figure 76a; similarly, the angle between two circles is zero if and only if they are tangent to one another); the tangent distance between the circles in Figure 76b, of which we stipulated that they are not tangent, is not zero (it is equal to AB for the first pair of circles in that figure and is undefined for the second pair).

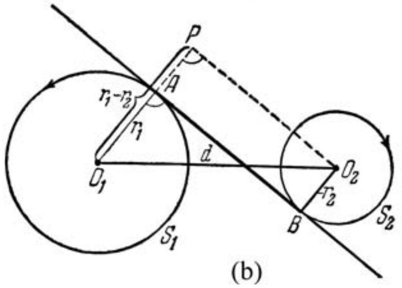
It is easy to see that *if the radii of two* directed *circles S_1 and S_2 are r_1 and r_2 and the distance between their centers is d then their tangent distance is*

$$t = \sqrt{d^2 - (r_1 - r_2)^2} \qquad (*)$$

(for proof apply the Pythagoras theorem to the triangle O_1O_2P in Figure



(a)



(b)

FIGURE 82

82; Figure 82a deals with the case when r_1 and r_2 have the same signs and Figure 82b with the case when these signs are different). This formula implies, in particular, that two circles S_1 and S_2 are tangent (that is, their tangent distance t is zero) if and only if

$$d^2 - (r_1 - r_2)^2 = 0,$$

that is, if the distance d between their centers is equal to the difference of their radii.

We now note the following important property of a dilatation.

C. *If a dilatation takes* (directed) *circles S_1 and S_2 to* (directed) *circles S_1' and S_2', then the tangent distance of S_1' and S_2' is equal to the tangent distance of S_1 and S_2* (Figure 83).

In fact, according to the definition of a dilatation, the radii r_1' and r_2' of S_1' and S_2' are $r_1 + a$ and $r_2 + a$, where a is the magnitude of the dilatation, and the distance d' between their centers is equal to the distance d between the centers of S_1 and S_2. Hence

$$\begin{aligned} t' &= \sqrt{d'^2 - (r_1' - r_2')^2} = \sqrt{d^2 - [(r_1 + a) - (r_2 + a)]^2} \\ &= \sqrt{d^2 - (r_1 - r_2)^2} = t, \end{aligned}$$

which was to be proved.

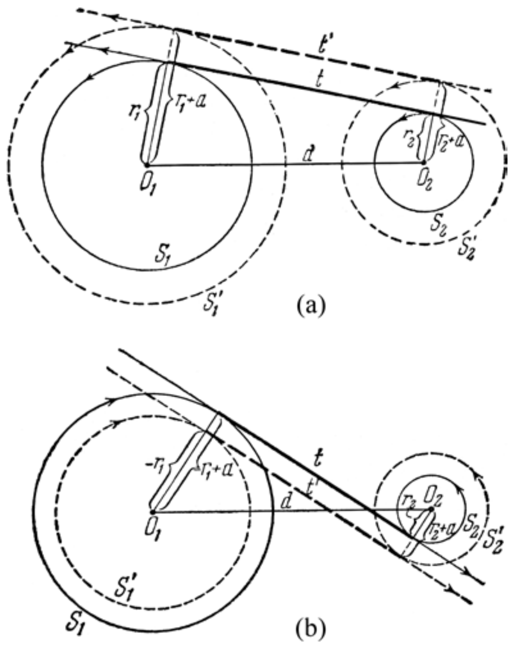


FIGURE **83**

Property C of a dilatation is a generalization of property B (which can be stated as follows: *If the tangent distance of two directed circles is zero, then the tangent distance of the transformed circles is also zero*).

60. Use a dilatation to solve problem 13(c) in NML 21.

61. Use a dilatation to solve the Apollonius problem (problem 26(a)).

62. (a) Use a dilatation to solve problem 50 in Section 4.

(b) Prove the converse of the propositions in problems 50 and 51: if the tangent distances of four circles S_1, S_2, S_3, and S_4 are connected by the relation

$$t_{12}t_{34} + t_{14}t_{23} = t_{13}t_{24}$$

(or $t_{12}t_{34} + t_{13}t_{24} = t_{14}t_{23}$, or $t_{14}t_{23} + t_{13}t_{24} = t_{12}t_{34}$), then these circles are tangent to some fifth circle, or to some line, or are concurrent.

[Here t_{12} denotes the tangent distance of S_1 and S_2 and t_{13}, t_{14}, t_{23}, t_{24}, and t_{34} have analogous meanings. Also, if, say, t_{12} and t_{13} are segments of like common tangents of S_1 and S_2 and of S_1 and S_3 (that is, both are segments of common outer or of common inner tangents), then t_{23} stands

for a segment of a common outer tangent of S_2 and S_3; if, on the other hand, t_{12} and t_{13} are segments of unlike common tangents (that is, one outer and one inner), then t_{23} stands for a segment of a common inner tangent of S_2 and S_3. In much the same way, if t_{12} and t_{14} are segments of like (unlike) common tangents, then t_{24} is a segment of a common outer (inner) tangent of S_2 and S_4, and so on.]

63. Use the result of the previous problem to prove the theorem in problem 11 in Section 1.

64. Prove that if four given circles S_1, S_2, S_3, and S_4 are tangent to three fixed circles Σ_1, Σ_2, and Σ_3, then these four circles are also tangent to some fourth circle Σ.

[In problem 64 we can replace some (or all) of the circles Σ_1, Σ_2, and Σ_3 by lines ("circles of infinite radius") or by points ("circles of zero radius"); the circle Σ may also turn out to be a line or a point.[1] If Σ_1, Σ_2, and Σ_3 are three lines, then Σ is the nine point circle of these three lines (see problem 63).]

The introduction of directed lines and circles is indispensable for the investigation of the transformations in this section and is also frequently useful in other matters (see, for example, pp. 20–21 and 30–31 in NML 8). A specific advantage of directed lines is that the angle between them is uniquely determined, and a specific advantage of directed circles is that they can have only one pair of common tangents. There are cases in which these properties of directed lines and circles yield essential simplifications. For example, a triangle with nondirected sides has six angle bisectors—three inner and three outer. They intersect three at a time in four points. Three of these points are the centers of escribed circles of the triangle and the fourth is the center of its inscribed circle (see Figure 84a). This fairly complicated configuration is significantly simplified if the sides of the triangle are directed lines, for then we have only three angle bisectors and a single escribed circle (Figure 84b).[2] [We leave it to the reader to explain the simplifications resulting from the introduction of directed lines in the fairly complicated theorems involved in the solution of problem 73(b) in NML 24.] Two directed circles have one center of similarity (rather than two!): an outer one in the case of circles with the same direction and an inner one in the case of circles with different directions. Three directed circles have three centers of similarity—one for each pair of circles. These three centers of similarity are collinear—they lie on the (unique!) axis of similarity of the three circles (Figure 85). (Compare Figure 85 with the far more involved Figure 19 in NML 8.) The formulation of the theorems in problems 50, 51, 62 and in subsequent problems is considerably simplified if the circles in these theorems are directed; also, we can dispense with the rather cumbersome clarifications following the statements of these theorems. There are many more instances of this kind.

We note one more fact of extreme importance for the present section. The properties of directed lines are not only far simpler than those of nondirected lines

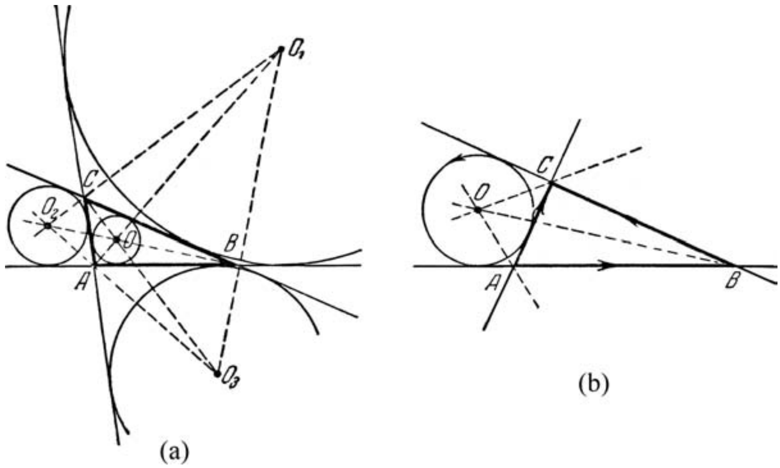


FIGURE 84

but they also resemble to a greater extent the properties of points. Thus the fact that *three directed lines are tangent to a unique directed circle* is analogous to the fact that three points lie on a unique circle; that *two directed circles have two, or one* (if they are tangent!) *or no common* (directed) *tangents* is analogous to the fact that two circles have two, one, or no common points, and so on.[3] This gives rise to the familiar parallelism ("duality") between the properties of points and of directed lines, a parallelism that is absent if the lines are not directed. To illustrate this issue more persuasively, we adduce a number of paired theorems in which the second theorem in a pair is obtained from its partner by replacing in the latter "point" by

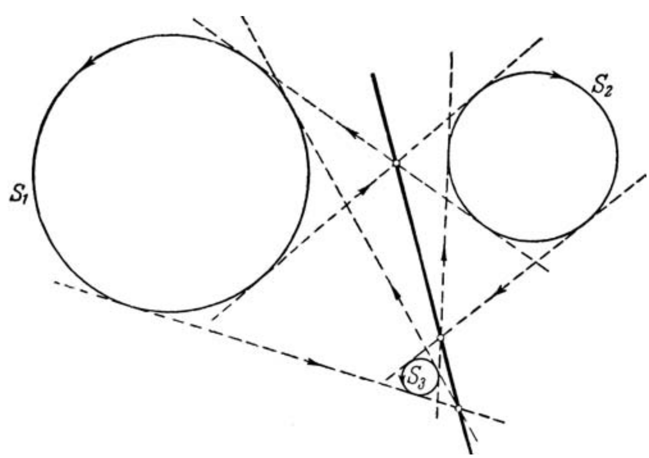


FIGURE 85

"directed line," "line" by "point," "circle" by "directed circle," "a point lies on a line" by "a directed line passes through a point," "a circle is tangent to a line" by "a directed circle passes through a point," and "a point lies on a circle" by "a directed line is tangent to a directed circle."[4]

I. If each of four circles S_1, S_2, S_3, and S_4 is tangent to two of its neighbors (S_1 to S_2 and S_4, S_2 to S_1 and S_3, and so on), then the four points of tangency lie on a single circle Σ (see problem 5 in Section 1).

I′. If each of four directed circles S_1, S_2, S_3, and S_4 is tangent to two of its neighbors, then the four directed common tangents of neighboring circles are tangent to a single directed circle Σ (see problem 73 below).

II. If A_1 and A_2 are the points of intersection of circles S_1 and S_2; B_1 and B_2—the points of intersection of S_2 and S_3; C_1 and C_2—the points of intersection of S_3 and S_4; and D_1 and D_2—the points of intersection of S_4 and S_1; and if A_1, B_1, C_1, D_1 lie on one circle, then A_2, B_2, C_2, D_2 also lie on one circle (see problem 6(b) in Section 1).

II′. If a_1 and a_2 are common (directed) tangents of circles S_1 and S_2; b_1 and b_2—common tangents of S_2 and S_3; c_1 and c_2—common tangents of S_3 and S_4; and d_1 and d_2—common tangents of S_4 and S_1; and if the directed lines a_1, b_1, c_1, d_1 are tangent to a directed circle, then a_2, b_2, c_2, d_2 are also tangent to a directed circle (see problem 67(a) below).

III. Let S_1, S_2, and S_3 be three circles which intersect in a point O; we will call the circle passing through the three other points of intersection of S_1, S_2, and S_3 the central circle of our three circles. Further, from four circles intersecting in O we can choose four triples of circles; the corresponding four central circles intersect in a single point—the central point of four circles (this assertion is equivalent to the theorem in problem 6(c) in Section 1). Similarly, from five circles intersecting in a point we can choose five quadruples of circles; the corresponding five central points lie on a single circle—the central circle of five circles, and so on (compare this with problem 7(a) in Section 1).

III′. Let S_1, S_2, and S_3 be three directed circles tangent to a directed line o; we will call the directed circle tangent to the three other (directed) common tangents of S_1, S_2, and S_3 the central circle of our three circles. Further, from four directed circles tangent to a directed line o we can choose four triples of circles; the corresponding four central circles are tangent to a directed line—the central line of four circles. Similarly, from five directed circles tangent to a directed line we can choose five quadruples of circles; the corresponding five central lines are tangent to a directed circle—the central circle of five circles, and so on (the reader should try to supply a proof!).

IV. Let S_1, S_2, and S_3 be three circles which intersect in a point O, and let A_1, A_2, and A_3 be three points on these three circles; then the circle Σ_1, passing through A_1, A_2, and the point of intersection of S_1 and S_2 different from O, and the circles Σ_2 and Σ_3 defined in a manner analogous to the definition of Σ_1, intersect in a point which we will call the directing point of our three circles (this assertion is equivalent to the theorem in problem 6(a) in Section 1). Further, let S_1, S_2, S_3, and S_4 be four

given circles intersecting in a point O, and on these circles four concyclic points A_1, A_2, A_3, and A_4; the four directing points of the four triples of circles which can be chosen from S_1, S_2, S_3, and S_4 lie on a single circle—the directing circle of our four circles. Similarly, from five circles S_1, S_2, S_3, S_4, and S_5 intersecting in a point, on which are given five concyclic points, we can choose five quadruples of circles; the corresponding five directed circles intersect in a point—the directing point of our five circles, and so on (compare this with problem 7(b) in Section 1).

IV′. Let S_1, S_2, and S_3 be three directed circles tangent to a directed line o, and let a_1, a_2, and a_3 be any three (directed) tangents to S_1, S_2, and S_3; then the directed circle Σ_1, tangent to a_1, a_2, and to a (directed) common tangent of S_1 and S_2 different from o, and the circles Σ_2 and Σ_3 defined in a manner analogous to the definition of Σ_1, are tangent to a single directed line—the directing line of our three circles (see problem 67(b) below). Further, let S_1, S_2, S_3, and S_4 be four given directed circles tangent to a directed line o and any (directed) tangents a_1, a_2, a_3, and a_4 of these circles all tangent to a single directed circle; then the four directing lines of the four triples of circles which can be chosen from S_1, S_2, S_3, and S_4 are tangent to a single directed circle—the directing circle of our four circles. Similarly, from five directed circles tangent to a directed line, with five of their (directed) tangents tangent to a single directed circle, we can choose five quadruples of circles; the corresponding five directing circles are tangent to a single directed line—the directing line of our five circles, and so on. (The reader should try to supply proofs!)

V. The circle passing through the midpoints of the sides of the triangle ABC (Figure 87a) is tangent to the inscribed and the three escribed circles of that triangle (see problem 11 in Section 1).

V′. The circle tangent to the three angle bisectors of the triangle ABC (Figure 87b; see Note 2 in the Notes to this section) is tangent to its circumscribed circle (and all four circles tangent to its angle bisectors have this property; by choosing different directions on the sides of the triangle ABC, and thus also different angle bisectors, we obtain a total of 16 triangles tangent to the circumscribed circle). [For proof note that in Figure 87b the points A, B, C are the feet of the altitudes of the

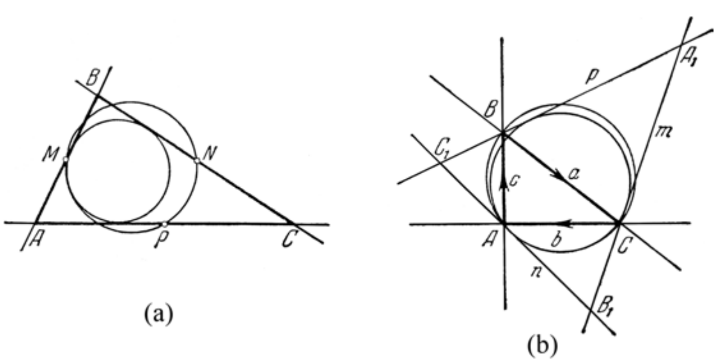


FIGURE 87

triangle $A_1B_1C_1$ and use the results of problem 17(a) in NML 21 and of problem 11 in Section 1.[5]]

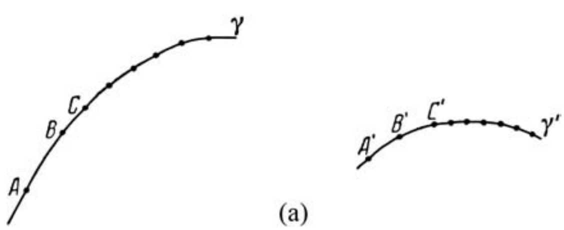
We could give many more such pairs of theorems.

We will see in the sequel that the parallelism between the properties of points and directed lines is truly profound.

Thus far we have viewed a dilatation as a transformation that takes a figure in the plane to a new figure. Now we will look at this transformation in a different way. In NML 8 we studied isometries, in NML 21—similarities, in NML 24—affine transformations, and in Sections 1–4 of this book—inversions. All these transformations are point-to-point transformations, that is, they take a point in the plane (or in a part of the plane, as in the case of projective transformations (see p. 52 in NML 24) and inversions (see p. 11)) to some new point. In Section 4 of NML 24 we studied polarities. These transformations interchange points and lines and thus are not point-to-point transformations. While dilatations are not point-to-point transformations, they are very different from polarities.

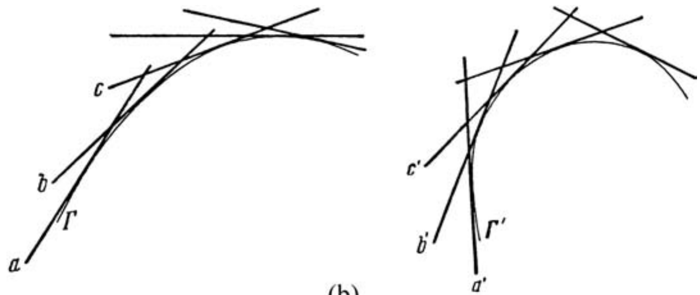
In Section 4 of NML 24 we discussed the principle of duality. According to this principle points and lines in the plane have—to a significant extent—the same status, in the sense that if we interchange the terms "point" and "line" in a theorem we obtain a new and true theorem. But if points and lines have equal status, then, in addition to point-to-point transformations, we should also study transformations that take lines to lines without necessarily taking points to points. Dilatations are an instance of just such transformations of the plane.

In the case of point-to-point transformations we viewed each geometric figure F as a set of points taken by the transformation to new points, and thus the figure F to a new figure F'. In particular, a point-to-point transformation takes a curve γ to some curve γ' (Figure 88a). In the case of line-to-line transformations we must view each geometric figure Φ as a set



(a)

FIGURE 88



(b)

FIGURE 88

of lines; the transformation takes lines to lines, and thus the figure Φ to a new figure Φ'. In particular, in this section we will view a curve as the set of its tangents; the line-to-line transformation takes each curve Γ to a new curve Γ' (Figure 88b).

In the case of line-to-line transformations it turns out to be convenient to work with directed lines (axes).[6] A transformation of the plane that takes each axis to an axis is called an *axial transformation*. Axial transformations can be said to correspond to point-to-point transformations under the duality principle.

In Section 1 we considered point-to-point circular transformations of the plane, that is, *point-to-point transformations of the plane that take circles* (including lines, viewed as circles of infinite radius) *to circles*. Dilatations are *axial transformations that take circles* (including points, viewed as circles of zero radius) *to circles*.[7] We will call such transformations *axial circular transformations*.[8] The definition of axial circular transformations is obtained from that of circular transformations by interchanging the terms "point" and (directed) "line." Comparison of the properties of point-to-point and axial circular transformations illustrates well the general principle of duality.

Dilatations are a special type of axial circular transformations. Similarities are another special type of such transformations (see NML 21). In the sequel we will encounter more complicated kinds of axial circular transformations.

Property A of dilatations implies that they take circles to circles (see Note 7). Property B of dilatations corresponds to an analogous property of inversions ("inversions take tangent circles to tangent circles"). We are about to explain what property of circular transformations corresponds under the principle of duality to property C of dilatations.

We look carefully at the meaning of the concept of tangent distance of two circles. In this section we view a circle not as a set of points but as the set of lines tangent to that circle. To define the tangent distance of two circles S_1 and S_2 we first single out their "common line" m, that is, their common tangent. Let A and B be two points of S_1 and S_2 on m (Figure 89a). The tangent distance of S_1 and S_2 is the length of the segment AB.

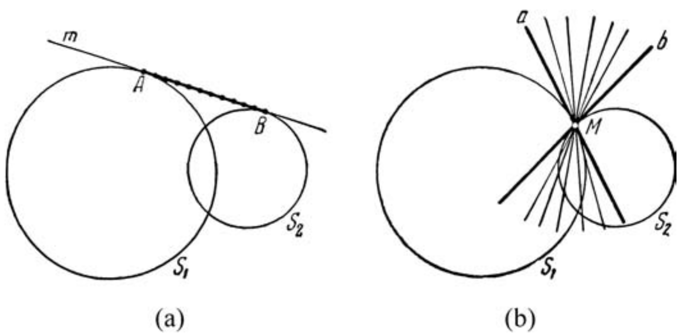


FIGURE 89

Now we interchange in our definition "point" and "line." We view S_1 and S_2 as sets of points; instead of their "common line" we consider their "common point," that is, their point of intersection M. Let a and b be "the lines of S_1 and S_2 passing through M," that is, the tangents to these circles at M (Figure 89b). Clearly, what corresponds under the principle of duality to the segment AB (the set of points on the line m between A and B) is the angle aMb (the set of lines through M between a and b), and what corresponds to the length of the segment AB is the magnitude of the angle aMb. But aMb is the angle between S_1 and S_2 (see Figure 3a). It follows that what corresponds under the principle of duality to the concept of tangent distance between two circles is the concept of the angle between two circles.

We saw earlier that inversion preserves the angle between two circles (see property C of inversion on p. 11). By the fundamental Theorem 2 in Section 4 (p. 73) it follows that the angle between two circles is preserved under all point-to-point circular transformations. To this property of circular transformations there corresponds the preservation of the tangent distance between two circles.[9]

B. Axial inversion

We will now describe a transformation called *axial inversion*, the dual of inversion (that is, the counterpart of inversion under the principle of duality).

We begin by proving a theorem which suggests how to define axial inversion.

Theorem 1. *Let S be a circle and l a line* (Figure 90; the line and the circle are directed). *Let M be a point on l outside S. Let a and b be the tangents from M to S. Then the product*[10]

$$\tan\frac{\widehat{la}}{2}\cdot\tan\frac{\widehat{lb}}{2}$$

depends only on S and l and is independent of the choice of M on l.

Proof. Let O be the center of S and r its radius. Let A and B be the points of tangency of a and b with S. Let P be the foot of the perpendicular from O to l and $d = OP$ the distance from the center of S to l (Figures 90a and 90b; in both figures the circle is positively directed and O is to the

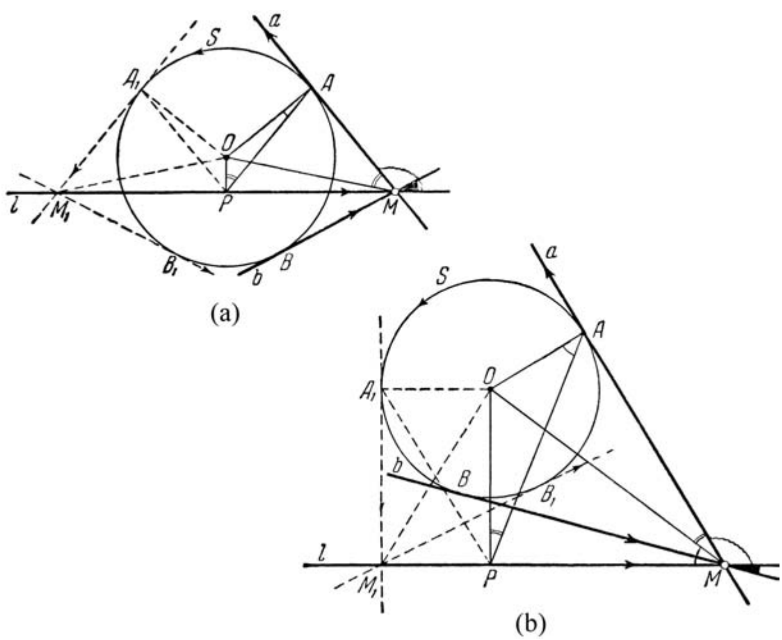


FIGURE 90

left of the directed line l). The tangent rule, applied to the triangle OAP, yields the equality

$$\frac{\tan\frac{\angle OAP-\angle OPA}{2}}{\tan\frac{\angle OAP+\angle OPA}{2}}=\frac{OP-OA}{OP+OA}.$$

Since two opposite angles in the quadrilateral $OAMP$ are right angles, we can circumscribe a circle about it. Hence $\angle OAP=\angle OMP$; $\angle OPA=\angle OMA$ (for these angles are subtended by the same arc). Now we use the fact that

$$\angle OMP+\angle OMA=\angle PMA;$$

$$\angle OMP-\angle OMA=\angle OMP-\angle OMB=\pm\angle PMB.$$

Since, in addition, $OP=d$ and $OA=r$, we have

$$\frac{\tan\frac{\angle PMB}{2}}{\tan\frac{\angle PMA}{2}}=\frac{|r-d|}{r+d}.$$

Figure 90 implies (note the directions of lines!) that

$$\angle PMB=\widehat{lb};\quad \angle PMA=180^\circ-\widehat{la}.$$

Hence

$$\tan\frac{\angle PMB}{2}=\tan\frac{\widehat{lb}}{2};$$

$$\tan\frac{\angle PMA}{2}=\tan\left(90^\circ-\frac{\widehat{la}}{2}\right)=\cot\frac{\widehat{la}}{2}=\frac{1}{\tan\frac{\widehat{la}}{2}}.$$

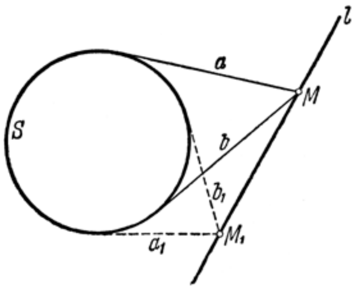
Thus

$$\tan\frac{\widehat{la}}{2}\cdot\tan\frac{\widehat{lb}}{2}=\frac{|r-d|}{r+d}. \qquad (*)$$

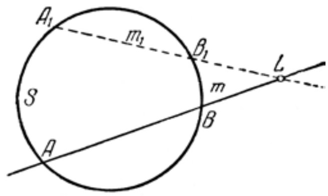
This shows that the product on the left is indeed independent of the choice of M on l (and dependent only on the choice of S and l).[11]

Now we will see what theorem is the dual of Theorem 1. In Theorem 1 we have the circle S and the line l (Figure 91a). Their duals are the circle S and the point L (Figure 91b). The dual of a point M on l is a line m passing through L; the duals of the tangents a and b from M to S are the points A and B of S on m, and the duals of the angles $\widehat{la}$ and $\widehat{lb}$ are the segments LA and LB. Now we state a well-known theorem of school geometry:

Theorem 1′. *Let S be a* (nondirected) *circle and L a point* (Figure 91b). *Let m be a line through L and A and B its points of intersection with S.*



(a)



(b)

FIGURE 91

Then the product

$$LA \cdot LB \tag{**}$$

depends on S *and* L *but not on the choice of the line through* L (see Theorems 1a and 1b in Section 3, p. 49).

This theorem is the dual of Theorem 1.

We called the product $LA \cdot LB$ the power of L with respect to S. It is reasonable to call the product $\tan \frac{\widehat{la}}{2} \cdot \tan \frac{\widehat{lb}}{2}$ the power of the (directed) line l with respect to the directed circle S.[12]

In Section 4 of NML 24 we called two theorems dual to one another if each of them could be obtained from the other by applying a polarity to the figure associated with the other. We will show that it is possible to deduce Theorem 1 from the well-known Theorem 1′, that is, we will show that these two theorems are dual to one another in the rigorous sense of Section 4 of NML 24.

A polarity with respect to the circle S takes Figure 91b to Figure 92. Assume that S and the lines l, a, b are directed; clearly, $\widehat{la} = \angle LOA$ and $\widehat{lb} = \angle LOB$ (property C of a polarity; see p. 82 in NML 24). Further, applying the so-called Mollweide formulas to the triangles LOA and LOB, we have:

$$\frac{LA}{LO+OA} = \frac{\sin \frac{\angle LOA}{2}}{\cos \frac{\angle LOA - \angle ALO}{2}}; \quad \frac{LB}{LO+OB} = \frac{\sin \frac{\angle LOB}{2}}{\cos \frac{\angle LOB - \angle BLO}{2}}.$$

Clearly, $\angle LAO = 180° - \angle LBO$, $\angle OLA = \angle OLB$ (Figure 92), or, if L is in the interior of S, $\angle LAO = \angle LBO$, $\angle OLA = 180° - \angle OLB$. Hence, in all cases,

$$\cos \frac{\angle LAO - \angle ALO}{2} = \sin \frac{\angle LBO + \angle BLO}{2} = \sin \frac{180° - \angle LOB}{2}$$

$$= \cos \frac{\angle LOB}{2};$$

$$\cos \frac{\angle LBO - \angle BLO}{2} = \cos \frac{\angle LOA}{2}.$$

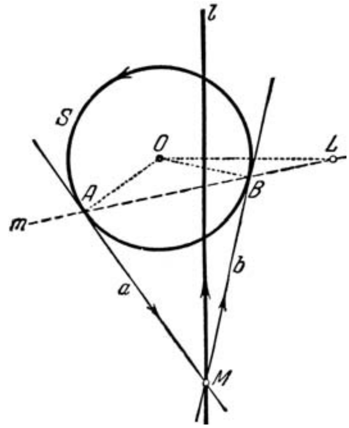


FIGURE 92

Denoting the distance OL by D and the radius of S by r, we have

$$\frac{LA}{D+r} = \frac{\sin \frac{\widehat{la}}{2}}{\cos \frac{\widehat{lb}}{2}}; \quad \frac{LB}{D+r} = \frac{\sin \frac{\widehat{lb}}{2}}{\cos \frac{\widehat{la}}{2}}.$$

Multiplying the two latter formulas we obtain

$$\frac{LA \cdot LB}{(D+r)^2} = \tan \frac{\widehat{la}}{2} \cdot \tan \frac{\widehat{lb}}{2}.$$

Since the product $LA \cdot LB$ depends only on L (Theorem 1′), the latter formula shows that the product $\tan \frac{\widehat{la}}{2} \cdot \tan \frac{\widehat{lb}}{2}$ depends only on l (Theorem 1).[13]

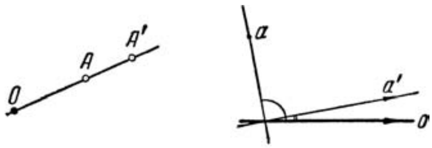
When deducing Theorem 1 from Theorem 1′ we assumed that S is positively directed and O is to the left of l (see Figures 92 and 90a). We leave it to the reader to discuss all other cases.

We recall the definition of inversion. An inversion with center O and power k is a (point-to-point) transformation which takes a point A to a point A' such the line AA' passes through O and

$$OA \cdot OA' = k$$

(Figure 93a); also, A and A' are on the same side of O. The analogy between Theorem 1 and Theorem 1′ suggests the following definition:

An axial inversion with (directed) *central line o and power k is an* (axial) *transformation which takes a* (directed) *line a to a* (directed) *line a′ such*



(a) (b)

FIGURE 93

that the point of intersection of a and a' is on o and

$$\tan\frac{\widehat{oa}}{2}\cdot\tan\frac{\widehat{oa'}}{2}=k$$

(Figure 93b); also, a and a' are directed to the same side of o.[14]

This definition is incomplete, for it does not show how to transform lines which do not intersect the central line o. For guidance note the following. If the angle $\widehat{oa}$ is small (that is, $\tan\frac{\widehat{oa}}{2}$ is small), then $\widehat{oa'}$ is close to 180° (that is, $\tan\frac{\widehat{oa'}}{2}$ is large). Drop perpendiculars MP and MP' from any point M on o to a and a' (Figure 94); let o be the point of intersection of a, a', and o. Then

$$MP = OM\sin\widehat{oa} = OM\frac{2\tan\frac{\widehat{oa}}{2}}{1+\tan^2\frac{\widehat{oa}}{2}},$$

$$MP' = OM\sin(180^\circ-\widehat{oa'}) = OM\frac{2\cot\frac{\widehat{oa'}}{2}}{1+\cot^2\frac{\widehat{oa'}}{2}}.$$

Now divide the first of these equations by the second. Since

$$1+\tan^2\frac{\widehat{oa}}{2}\approx 1 \quad\text{and}\quad 1+\cot^2\frac{\widehat{oa'}}{2}\approx 1$$

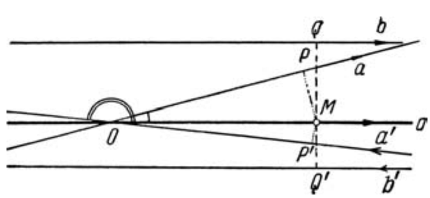


FIGURE 94

(tan $\frac{\widehat{oa}}{2}$ and cot $\frac{\widehat{oa'}}{2}$ are small) and

$$\tan\frac{\widehat{oa}}{2} : \cot\frac{\widehat{oa'}}{2} = \tan\frac{\widehat{oa}}{2}\cdot\tan\frac{\widehat{oa'}}{2} = k,$$

it follows that

$$\frac{MP}{MP'} \approx k.$$

This being so, it is natural to take for granted that *an axial inversion of power k takes a line b parallel to the central line o and at a distance MQ from o to a line b' counterparallel*[15] *to o and at a distance MQ' from o such that* $\frac{QM}{MQ'} = k$ (Figure 94). Conversely, the image of b' is b.

An axial symmetry can also be defined geometrically. Let o be the central line of an axial symmetry, Σ a circle tangent to a line l and to its image l' (such a circle is called the directing circle of the axial symmetry), and a_0 and a_0' two (directed) tangents to Σ which meet on o (Figure 95). In view of Theorem 1, we have, obviously,

$$\tan\frac{\widehat{oa_0}}{2}\cdot\tan\frac{\widehat{oa_0'}}{2} = \tan\frac{\widehat{ol}}{2}\cdot\tan\frac{\widehat{ol'}}{2} = k,$$

where k is the power of the axial inversion. This leads to the following definition of an axial inversion.

An axial inversion with central line o and directing circle Σ is an axial *transformation which takes every* (directed) *line a to a* (directed) *line a' such that a and a' meet on o, and the tangents a_0 and a_0' to Σ, parallel to a and a', also meet on o* (Figure 95). [This definition also fails to describe the effect of the transformation on lines that do not intersect the central line

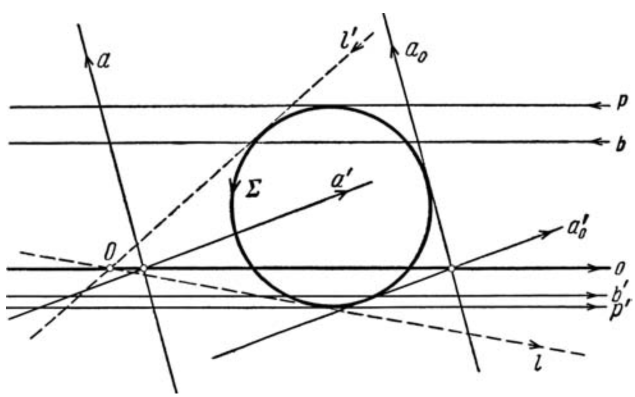


FIGURE 95

o. This calls for the following supplementary statement: *An axial inversion interchanges tangents p and p' to the directing circle Σ; it takes a line b parallel to p to a line b' parallel to p' and such that the distances of b and b' from o are proportional to the distances of p and p' from o*.]

If an axial inversion is given by its central line o and its directing circle Σ, we can easily construct the image a' (or b') of a given line a (or b). To this end we construct successively a_0, a'_0, and a' (or p, p', and b'). We note that since a_0 and a'_0 must be directed to the same side of o, Σ intersects o.

It is clear that the directing circle Σ of a given axial inversion is not uniquely determined: in fact, one can inscribe infinitely many circles in the angle lOl' in Figure 95 and one can choose the lines l and l' in different ways.

We saw in Section 1 of NML 24 that it is sometimes convenient to assign signs to segments. Similarly, we sometimes assign signs to angles formed by (distinct and directed) lines; namely, we say that angles $\widehat{ab}$ and $\widehat{cd}$ have the same signs if their directions—from a to b and from c to d—coincide (Figure 96a), and different signs if these directions are different (Figure 96b). Of course, the order of the rays of an angle is essential; according to our definition, the angles $\widehat{ab}$ and $\widehat{ba}$ have opposite signs.

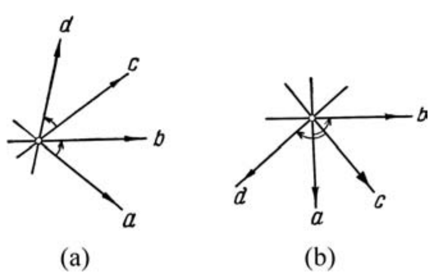


FIGURE 96

If we take into consideration the signs of angles, then we must take the power $LA \cdot LB$ of a point L with respect to a circle S as positive if L is outside S (see Figure 91b; here LA and LB have the same directions) and as negative if L is inside S (see, for example, Figures 50a and 50b on p. 49). If r is the radius of S and d is the distance of L from the center of S, then the power of L with respect to S (including sign!) is

$$d^2 - r^2 = (d - r)(d + r)$$

(see p. 50). Similarly, if we take into consideration the signs of angles, then the power of a (directed) line l with respect to a (directed) circle S, equal to the product $\tan\frac{\widehat{la}}{2}\cdot\tan\frac{\widehat{lb}}{2}$, will be positive if l intersects S (see Figure 90a; here the directions of the angles $\widehat{la}$ and $\widehat{lb}$ coincide) and negative if l does not intersect S (Figure 90b). If r is the (positive or negative) radius of S and d is the (positive or negative) distance from the center of S to the line l, then the power of l with respect to S (including the sign!) is equal to

$$\frac{r-d}{r+d}$$

(see the proof of Theorem 1 on pp. 100–101; we suggest that the reader examine the various cases involved).

If we take into consideration the signs of segments, then the power k of (an "ordinary") inversion can be positive or negative; the latter implies that the signs of the segments OA and OA' are different, which means that A and A' are on different sides of O (Figure 97a). Note that an inversion of negative power $-k$ is equivalent to an inversion of positive power k followed by a reflection in the center of inversion O. Similarly, if we take into consideration the signs of angles, then the power of an axial inversion can be positive or negative; the latter means that the angles $\widehat{oa}$ and $\widehat{oa'}$ have different signs, that is, that the lines a and a' are directed to different sides of the line o (Figure 97b). Clearly, an axial inversion of negative power

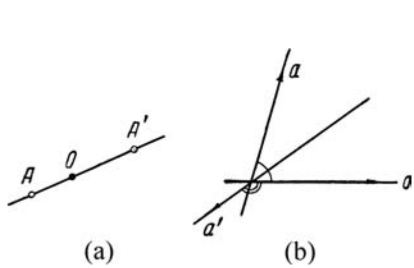


FIGURE 97

$-k$ is equivalent to an axial inversion of positive power k followed by a reflection in the central line o. One can give a geometric description of an axial inversion of negative power that is equivalent to the definition of an axial inversion of positive power, except that in this case *the directing circle* Σ *will not intersect the central line* o (Figure 98).

We note that *an axial inversion* (of positive or negative power) *interchanges* (directed) *lines* (that is, if it takes a to a' then it takes a' to a).

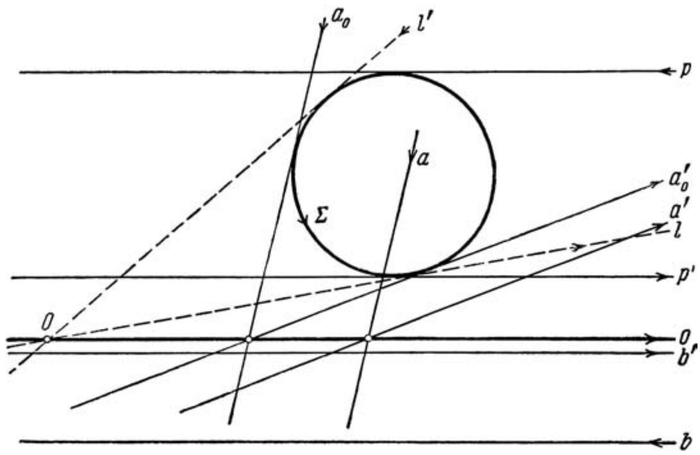


FIGURE 98

We now list and discuss the most important properties of axial inversions.

A. *An axial inversion takes parallel lines to parallel lines* (Figure 99).

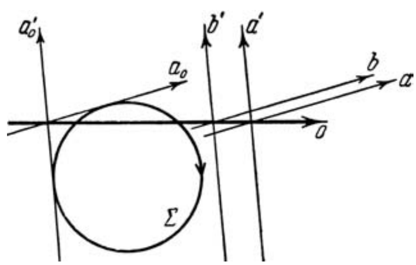


FIGURE 99

Property A follows directly from the the first and second definitions of an inversion.

B. *An axial inversion takes a* (directed) *circle or a point to a* (directed) *circle or a point* (compare this with property C of an ordinary inversion in Section 1, p. 11).

Since an axial inversion of negative power is equivalent to an axial inversion of positive power followed by a reflection in a line, it is enough to prove property B for an axial inversion of positive power. This means that the directing circle Σ of inversion intersects the central line o in points Q_0 and R_0 (Figure 100). Further, let Z_0 be the point of intersection of the

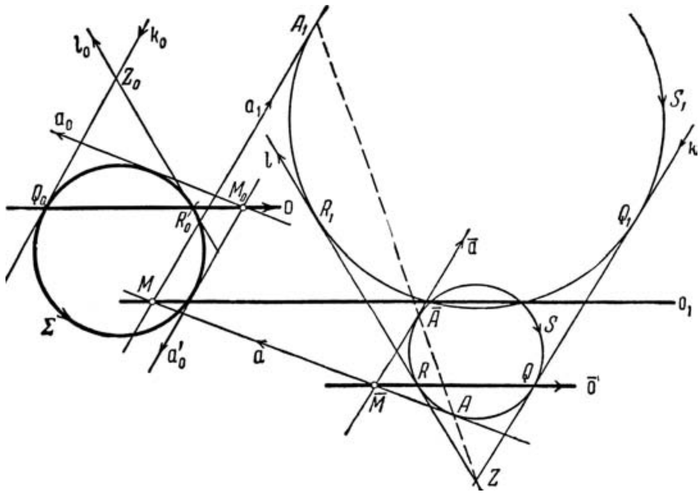


FIGURE 100

tangents k_0 and l_0 to Σ at Q_0 and R_0,[16] S a directed circle (or a point), a a tangent to S, a_0 a tangent to Σ parallel to a, a_0' another tangent to Σ drawn from the point M_0 of intersection of a_0 and o, and $\overline{a}$ a tangent to S parallel to a_0'.

Similarity considerations imply that $\overline{M}$, the point of intersection of a and $\overline{a}$, is on the line $\overline{o}$ parallel to o and disposed with respect to S in the same way as o is disposed with respect to Σ (in other words, the distance of the center of S from $\overline{o}$ is related to the radius of S as the distance of the center of Σ from o is related to the radius of Σ). We denote the points of intersection of S and $\overline{o}$ by Q and R and the point of intersection of the tangents k and l to S at Q and R by Z (clearly, k and l are parallel to k_0 and l_0 respectively).

We consider an arbitrary (directed) circle S_1 tangent to k and l; let a_1 be tangent to S_1 and parallel to $\overline{a}$, and let M be the point of intersection of a and a_1. We claim that M lies on the radical axis o_1 of S and S_1 (see Section 3, p. 49). Let A, $\overline{A}$, and A_1 be the points at which $a, \overline{a}$, and a_1 are tangent to the appropriate one of the circles S and S_1. Obviously, S_1 is centrally similar to S with center of similarity at Z, and $\overline{A}$ and A_1 are corresponding points of these circles. Hence $\overline{A}A_1$ passes through Z. On the other hand, $\overline{o}$ is the polar of Z with respect to S (see Section 4 in NML 24, p. 66f.); since $\overline{o}$ passes through $\overline{M}$, the polar of $\overline{M}$—that is, the line $A\overline{A}$—passes through Z. It follows that A, $\overline{A}$, and A_1 lie on a line passing through Z. Now the

similarity of the triangles $\overline{M}A\overline{A}$ and MAA_1 implies that

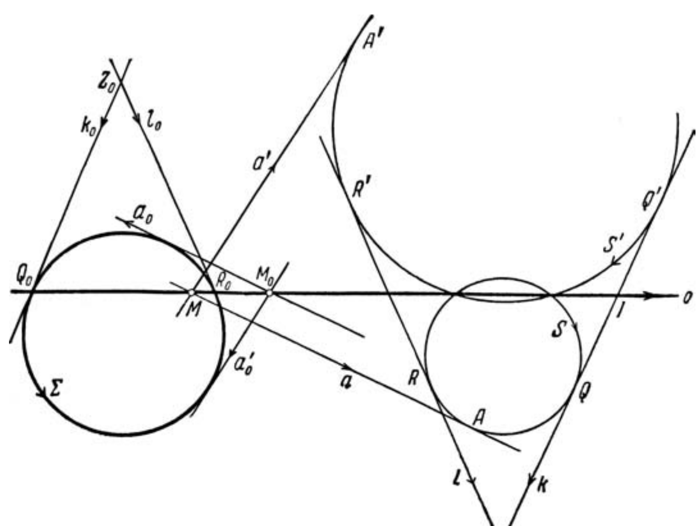
$$\frac{MA}{MA_1} = \frac{\overline{M}A}{\overline{M}\,\overline{A}}.$$

Since, obviously, $\overline{M}A = \overline{M}\,\overline{A}$, the latter relation implies that

$$MA = MA_1.$$

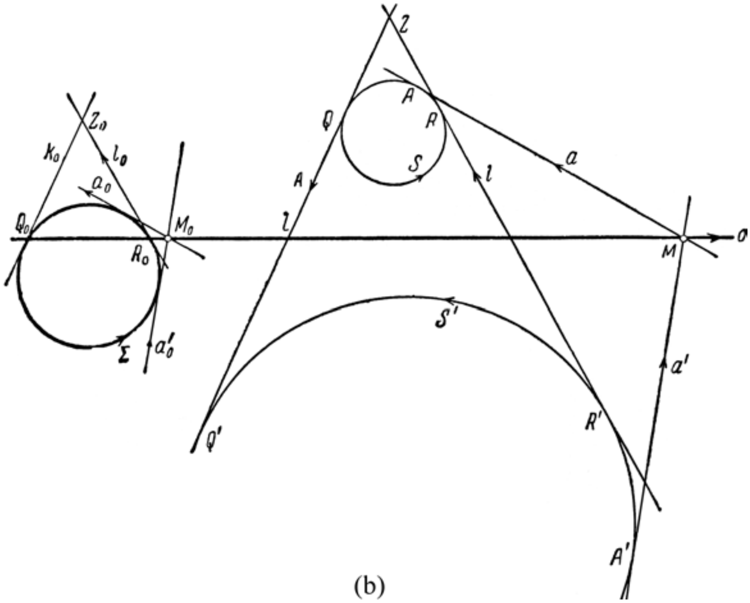
But this means that M lies on the radical axis o_1 of S and S_1. Note also that since the line of centers of S and S_1—that is, the bisector of $\angle OZR$—is perpendicular to $\overline{o}$ and o, it follows that $o_1 \parallel o$.

So far we assumed that S_1 was an arbitrary circle inscribed in $\angle QZR$. Now we choose this circle in a special way, namely, we require that the segment QQ_1 of k between its points of tangency with S and S_1 be halved by o (Figure 101). Then the radical axis o_1 of S and S_1 coincides with o (this follows from the fact that the point I of intersection of k and the central line o lies on the radical axis of S and S_1: $IQ = IQ_1$). But if a_1 and a intersect on the central line o, then it follows that a_1 coincides with a', the image of a under the axial inversion (see the definition of an axial inversion on p. 105). This means that S_1 (which we denote in this case by



(a)

FIGURE 101



(b)

FIGURE 101

S') is the image of S under the axial inversion. This completes the proof of property B of an axial inversion.[17]

The method of construction of the circle S', the image of the given circle S, follows directly from the proof of property B: *the circle S' is tangent to the tangents k and l of the circle S, which are parallel to the tangents k_0 and l_0 to the directing circle Σ at the points of intersection of Σ with the central line o; also, the central line o halves the segments QQ' and RR' of the common tangents of S and S'*. If S intersects o (Figure 101a) then, when constructing the circle S', we can use as a starting point the fact that o is a common chord of S and S' (this follows from the fact that o is the radical axis of S and S').

This construction implies certain inferences. It is easy to show that *if the circle S does not intersect the the central line of the axial inversion, then we can choose the directing circle Σ so that the image circle S' is a point* ("a circle of radius zero"); *and if S intersects o, then we can choose Σ so that the center of S' is on o*. In fact, suppose that S does not intersect o. Let P be the foot of the perpendicular from the center O of S to o; PT the tangent from P to S; and S' a point on the extension of OP beyond P such that $PS' = PT$ (Figure 102a). It is easy to see that in this case the

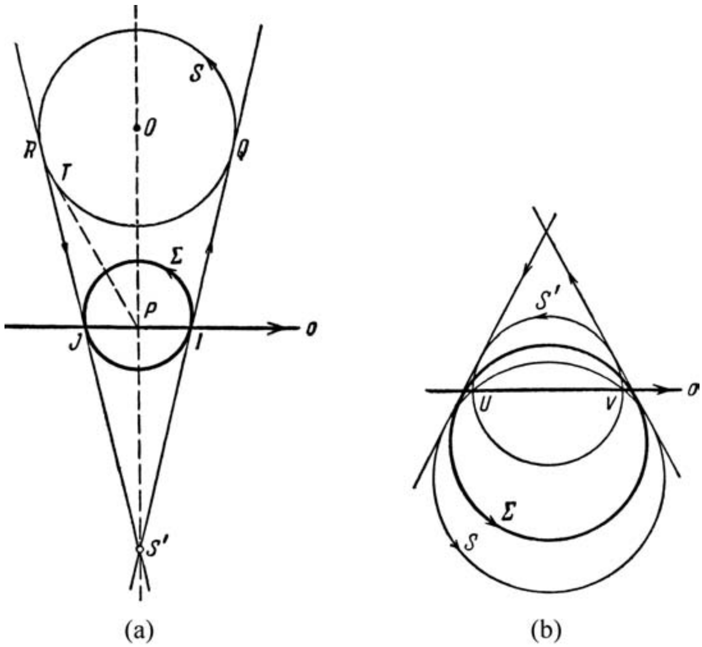


FIGURE 102

radical axis of the point S' (a "circle of zero radius") coincides with o (for $PT = PS'$ implies that P is on the radical axis). Hence *if the directing circle* Σ *of an axial inversion is such that the tangents to this circle at its points of intersection with the axis o are parallel to the tangents* $S'Q$ *and* $S'R$ *drawn from* S' *to* S (for example, if Σ is tangent to the segments $S'Q$ and $S'R$ at their midpoints I and J; Figure 102a), *then the axial inversion takes* S *to the point* S'. *And if* S *intersects o at points* U *and* V *then, to transform* S *to the circle* S' *with diameter* UV (S' has the same direction as S), *we need only require that the tangents to the directing circle* Σ *at its points of intersection with the central line o be parallel to the common tangents of* S *and* S' (for example, that Σ be tangent to these common tangents at the points in which they intersect o; Figure 102b).

Further, suppose that an axial inversion takes a circle S (which does not intersect the central line o) to a point S', and m and n are tangents to S from (an arbitrary) point M on o, and that S_1 is (an arbitrary) circle tangent to m and n (Figure 103).

The images m' and n' of m and n under our axial inversion pass through M and S', and so can differ only by direction; the image S_1' of S_1 under

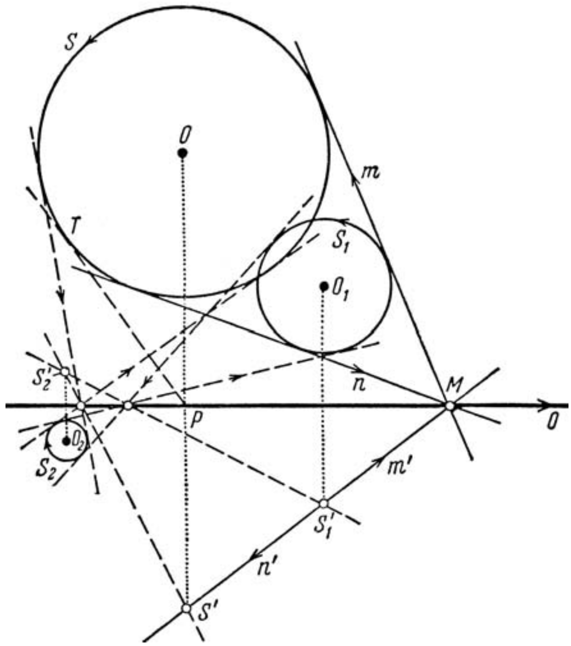


FIGURE **103**

the axial inversion is tangent to m' and n'. Obviously, only a point (and not a circle of nonzero radius) can "be tangent" to two lines that coincide in terms of position but have opposite directions. It follows that the image of S_1 under our axial inversion is also a circle. We conclude that *a suitably chosen axial inversion* (or dilatation) *takes two circles S and S_1 with two common tangents to two points*; for this it suffices to require that the central line o of the axial inversion should pass through the point of intersection of m and n and that the image of S should be a point (if $m \parallel n$, then S and S_1 can be transformed into points by a dilatation). Also, *a suitably chosen axial inversion will take three circles S, S_1, and S_2 whose axis of similarity o* (see p. 29 in NML 21)[18] *does not intersect these circles into three points*; for this we need only use an axial inversion with central line o which takes S to a point (Figure 103).[19]

We now give an algebraic proof of property B of axial inversion based on the first definition of this transformation. Suppose that a tangent a to a circle S intersects the central line o at M; that the radius of S is r; and that the distance from the center O of S to o is d (Figure 104). We drop perpendiculars OP and OQ from O to o and a. Let K be the point of intersection of a and OK parallel to o. Let P' be the foot

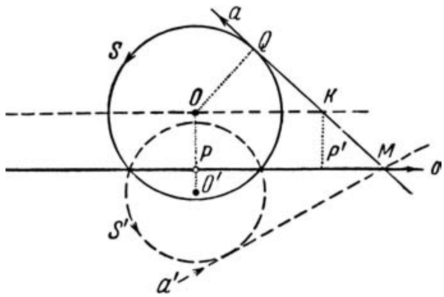


FIGURE 104

of the perpendicular from K to o. Then, clearly,[20]

$$MP = MP' + P'P.$$

Consider the triangles $MP'K$ and KOQ. We have

$$MP' = \frac{KP'}{-\tan\widehat{oa}} = -\frac{d}{\tan\widehat{oa}}, \quad PP' = KO = \frac{OQ}{\sin\widehat{oa}} = \frac{r}{\sin\widehat{oa}}.$$

Hence

$$MP = -\frac{d}{\tan\widehat{oa}} + \frac{r}{\sin\widehat{oa}} = -d : \frac{2\tan\frac{\widehat{oa}}{2}}{1-\tan^2\frac{\widehat{oa}}{2}} + r : \frac{2\tan\frac{\widehat{oa}}{2}}{1+\tan^2\frac{\widehat{oa}}{2}},$$

or, finally,

$$MP = A\tan\frac{\widehat{oa}}{2} + B\cot\frac{\widehat{oa}}{2}, \tag{$*$}$$

where

$$A = \frac{d+r}{2}, \quad B = \frac{-d+r}{2}.$$

Now we apply an axial inversion of power k. Then a goes over to a line a' which intersects o at (the same point) M such that

$$\tan\frac{\widehat{oa}}{2}\tan\frac{\widehat{oa'}}{2} = k,$$

that is,

$$\tan\frac{\widehat{oa}}{2} = k\cot\frac{\widehat{oa'}}{2}, \quad \cot\frac{\widehat{oa}}{2} = \frac{1}{k}\tan\frac{\widehat{oa'}}{2}.$$

Hence for the new line a' we have:

$$MP = Ak\cot\frac{\widehat{oa'}}{2} + \frac{B}{k}\tan\frac{\widehat{oa'}}{2},$$

or

$$MP = A'\tan\frac{\widehat{oa'}}{2} + B'\cot\frac{\widehat{oa'}}{2}, \tag{$**$}$$

where

$$A' = \frac{B}{k}, \quad B' = Ak.$$

From $(*)$ and $(**)$ we conclude that a' is tangent to the circle S'. Its radius is r'. The distance $O'P = d'$ from the center O' of S' to the axis o is determined from the relations

$$\frac{d' + r'}{2} = A', \quad \frac{-d' + r'}{2} = B'$$

(Figure 104). And this is what we wished to prove.

We note that the formulas we have obtained imply that

$$r' = A' + B' = \frac{B}{k} + Ak = \frac{-d + r}{2k} + \frac{(d + r)k}{2},$$
$$d' = A' - B' = \frac{B}{k} - Ak = \frac{-d + r}{2k} - \frac{(d + r)k}{2}.$$

Hence for the image of S' to be a point it is necessary and sufficient that

$$r' = \frac{-d + r}{2k} + \frac{(d + r)k}{2} = 0,$$

or

$$k^2 = \frac{d - r}{d + r}; \quad k = \sqrt{\frac{d - r}{d + r}} = \frac{\sqrt{d^2 - r^2}}{d + r}.$$

It follows that if a (directed) line does not intersect a (directed) circle S, that is, if $d^2 - r^2 > 0$, then one can take S to a point by an axial inversion with central line o. For this it suffices to choose the power k of the inversion equal to

$$k = \frac{\sqrt{d^2 - r^2}}{d + r}.$$

[Here the images of all circles with $\frac{d-r}{d+r} = k^2$, or $\frac{d}{r} = \frac{1+k^2}{1-k^2}$, are points. This family of circles is characterized by the fact that the center of similarity of any two of them lies on the axis o.]

For the center of an image circle to lie on the central line o it is necessary and sufficient that

$$d' = \frac{-d + r}{2k} - \frac{(d + r)k}{2} = 0,$$

or

$$k^2 = \frac{-d + r}{d + r}; \quad k = \sqrt{\frac{-d + r}{d + r}} = \frac{\sqrt{r^2 - d^2}}{r + d}.$$

Hence *if S intersects o*, that is, if $r^2 - d^2 > 0$, then S can be taken to a circle with center on o; for this it suffices to choose the power k of the inversion equal to

$$k = \frac{\sqrt{r^2 - d^2}}{r + d}.$$

C. *An axial inversion preserves the tangent distance of two circles* (compare this with property C of an ordinary inversion; see Section 1, p. 11).

First we assume that one of two circles is a point M (a "circle of zero radius") on the axis of inversion o (Figure 105a). Under the inversion M will go over to itself, and the tangent distance MA between M and the circle S will be equal to the tangent distance MA' between M and the image S' of S (this is so because o is the radical axis of S and S'; see the proof of property B of axial inversion).

Now let S_1 and S_2 be two circles. Let AB be their common tangent which intersects the axis of inversion o at M. Let S_1' and S_2' be the images of S_1 and S_2 under the axial inversion and let $A'B'$ be their common tangent (Figure 105b).

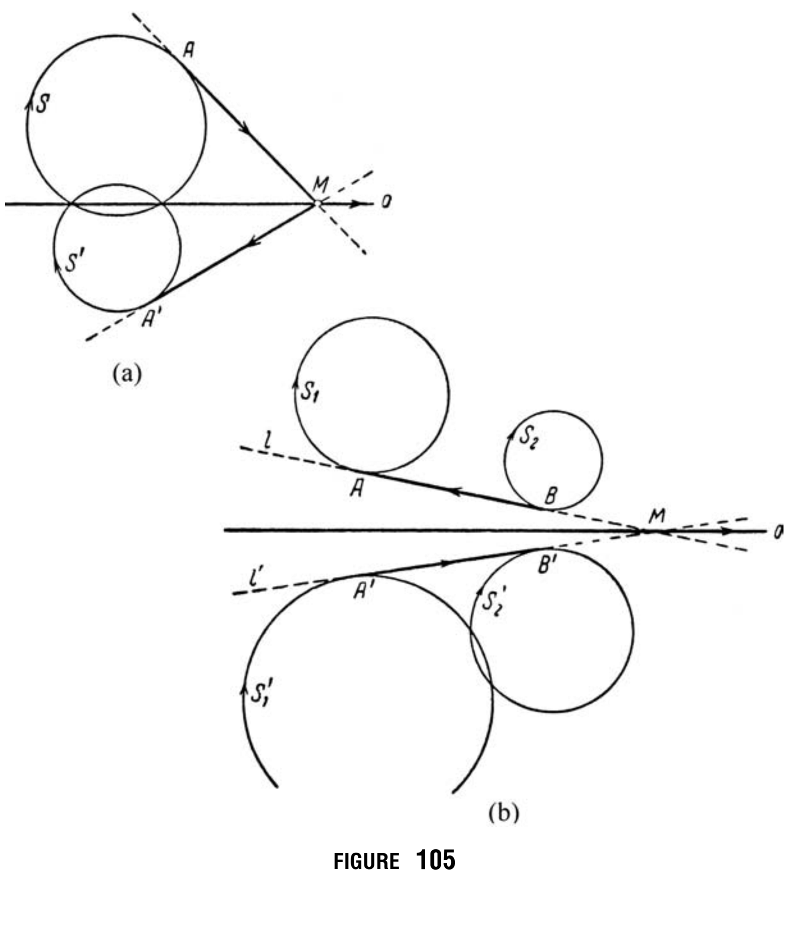


FIGURE 105

From what has just been proved it follows that

$$AM = A'M, \quad BM = B'M.$$

Hence

$$AB = A'B',$$

which is what was to be proved.

Property C of axial inversion implies, in particular, that *an axial inversion takes two tangent circles to two tangent circles.*

In the case of ordinary inversion the angle between circles does not change size but does change direction (see p. 13). Similarly, it makes sense to say that in the case of axial inversion the tangent distance between two circles does not change size but does change direction. Specifically, if AB is the tangent distance between circles S_1 and S_2 and the direction of the segment AB from A to B is opposite to the direction of the common tangent l of the circles of which this segment is a part, then the tangent distance $A'B'$ of the transformed circles S_1' and S_2' is equal to the distance AB, but the direction of the segment $A'B'$ from A' to B' will coincide with the direction of l', the image line of l (Figure 105b).

In analogy to the definition of the tangent distance of two circles we can define the tangent distance of two curves γ_1 and γ_2 as the length of their common tangent between the points of tangency (Figure 106; for the notion of a tangent to an arbitrary curve see p. 13). One can show that *if an axial inversion takes curves γ_1 and γ_2 to curves γ_1' and γ_2', then the tangent distance between γ_1' and γ_2' is equal to the tangent distance between γ_1 and γ_2* (in other words, an axial inversion preserves the tangent distance between two curves).[20] We won't discuss this property of axial inversion because we do not use it.

65. (a) Use an axial inversion to solve the problem of Apollonius (see problem 26(a) in Section 2, p. 36) in the case where the axis of similarity of the given circles S_1, S_2, and S_3 does not intersect any of them.

(b) Use an axial inversion to prove the theorem in problem 50 (p. 69) in the case where the axis of similarity of the circles S_1, S_2, and S_3 does not intersect any of them.

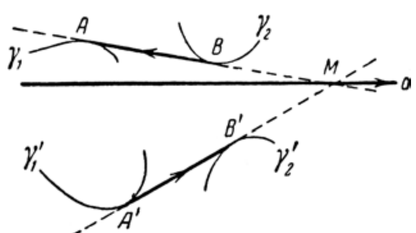


FIGURE 106

66. Use the properties of axial inversion to deduce the theorem of Brianchon (see problem 47 on p. 63 in NML 24) from the fact that the radical axes of three circles taken two at a time are concurrent (see Section 3, p. 49).

67. (a) We are given four circles S_1, S_2, S_3, and S_4. Let a_1, a_2 be the common tangents of S_1 and S_2; b_1, b_2 the common tangents of S_2 and S_3; c_1, c_2 the common tangents of S_3 and S_4; and d_1, d_2 the common tangents of S_4 and S_1. Show that if a_1, b_1, c_1, and d_1 are tangent to a circle Σ, then a_2, b_2, c_2, and d_2 are tangent to a circle $\overline{\Sigma}$ (Figure 107a).[21]

(b) Let S_1, S_2, and S_3 be three given circles and let a_1, a_2, and a_3 be tangent to these circles. Let b_3, b_2, and b_1 be common tangents of the pairs S_1, S_2; S_1, S_3; and S_2, S_3. Finally, let Σ_1, Σ_2, and Σ_3 be tangent to the triples a_2, a_3, b_1; a_1, a_3, b_2; and a_1, a_2, b_3 (Figure 107b).[22] Show that if

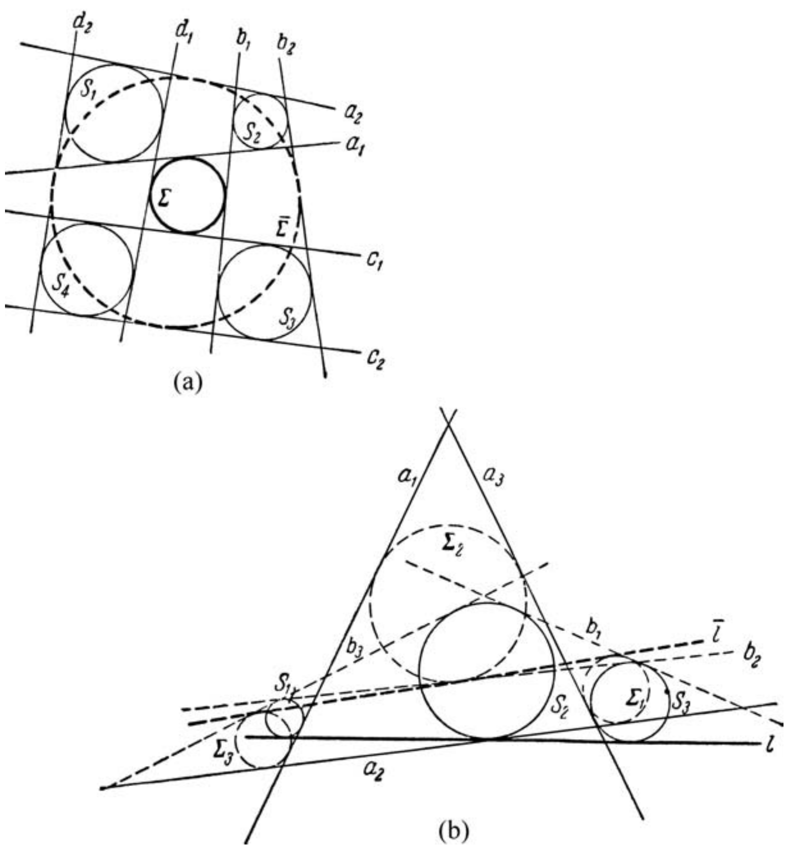


FIGURE 107

S_1, S_2, and S_3 are tangent to a line l (different from b_1, b_2, and b_3), then Σ_1, Σ_2, and Σ_3 are tangent to a line $\bar{l}$ (different from a_1, a_2, and a_3).

68. Let A, B, C be three circles and let a, b, c be the tangent distances of these circles taken two at a time. Let D be a circle tangent to the common tangents K_1L_1 and K_2L_2 of the circles A and B at the midpoints of the segments K_1L_1 and K_2L_2 (Figure 109).[22] Find the tangent distance x of the circles C and D.

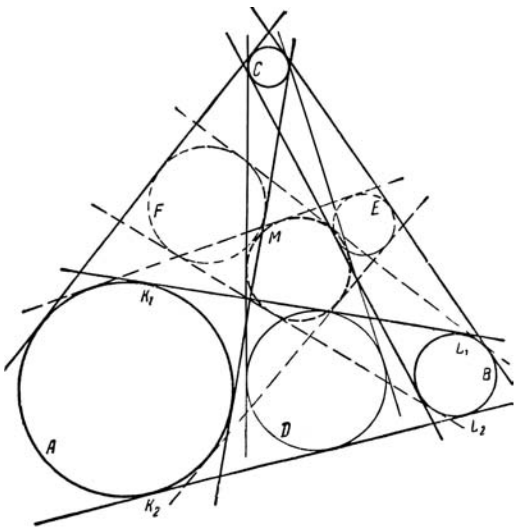


FIGURE 109

69. Let A, B, C be three circles. Let D, E, F be circles tangent to the common tangents of the pairs of circles A, B; B, C; and C, A at the midpoints of the segments determined by the points of tangency (Figure 109).[22] Show that

(a) the common tangents of the three pairs of circles C, D; A, E; and B, F are tangent to a circle M;

(b) the points of tangency of the common tangents of the three pairs of circles C, D; A, E; and B, F with the circle M divide the segments of these tangents between corresponding circles in the ratio 2:1 (beginning at the circles C, A, and B).[23]

70. The cross ratio of four (directed) lines is defined by the expression

$$\frac{\sin\frac{\widehat{ac}}{2}}{\sin\frac{\widehat{bc}}{2}} : \frac{\sin\frac{\widehat{ad}}{2}}{\sin\frac{\widehat{bd}}{2}}$$

(compare this with the definition of the cross ratio of four points on p. 35 in NML 24). Show that an axial inversion preserves the cross ratio of four lines (compare this with property D of an "ordinary" inversion on p. 63).

71. Find the locus of points which a given sequence of axial inversions takes to points.

72. Circumscribe about a given circle S an n-gon whose vertices lie on n given lines $l_1, l_2, \ldots, l_n$.

Problem 72 is stated in another context as problem 84(b) on p. 97 in NML 24.

We will now prove a theorem which is sometimes helpful when using axial inversions in the solution of problems.

Theorem 2. *Using axial inversions it is always possible to take two* (directed) *circles, or a* (directed) *circle and a point, to two points, or to a point and a circle passing through it, or to two circles that differ only by direction* (compare this theorem with Theorem 2 in Section 1 on p. 22).

Proof. We saw earlier that using an axial inversion one can take two circles with two common tangents to two points (see p. 113). Further, since one can take any circle by an axial inversion to a point, one can take two tangent circles (circles with a unique common tangent) by an axial inversion to a point and a circle passing through it—for this we need only take one of these two circles to a point. Thus it remains to show that it is possible to take two circles without a common tangent (Figure 110) by an axial inversion to two circles S_1' and S_2' that differ only by direction.

If o is the central line of the required axial inversion, then o is the radical axis of the (nondirected) circles S_1 and S_1' and the radical axis of the (nondirected) circles S_2 and S_1' (see the proof of property B of axial inversion). This means that the power of any point of o with respect to S_1 and S_1' is the same as its power with respect to S_2 and S_1'. But then the power of any point of o with respect to S_1 is the same as its power with respect to S_2, that is, o is the radical axis of S_1 and S_2.

Now let p_1 be a tangent to S_1 parallel to o and p_2 be a tangent to S_2 counterparallel to o.[24] Our axial inversion takes p_1 and p_2 to tangents p_1' and p_2' of S_1' and S_2'. Since S_1' and S_2' differ only by direction, p_1' and p_2'

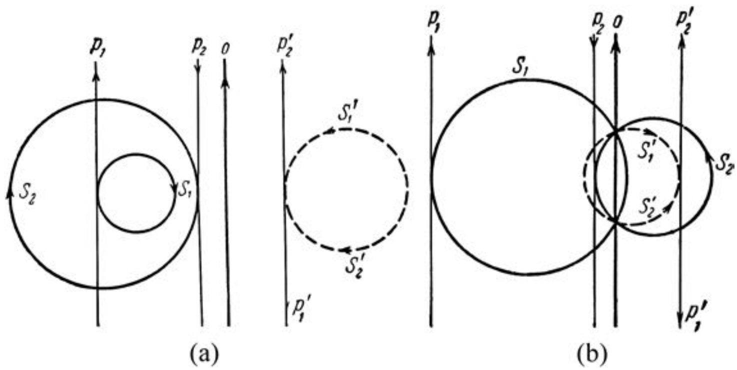


FIGURE 110

also differ only by direction. We denote the distances (positive or negative, according to the rule that determines the sign of the distance from a point to a line) from the lines p_1, p_2, and p_1' to o by d_1, d_2, and d. If k is the power of the axial inversion, then[25]

$$\frac{d_1}{d} = -k, \quad \frac{d}{d_2} = -k$$

(see p. 105; keep in mind that p_1 and p_2' are parallel to o). Multiplying these equalities we obtain

$$\frac{d_1}{d_2} = k^2,$$

that is, the square of the power of the inversion is equal to the ratio of the distances from p_1 and p_2 to o. Also, the ratio $\frac{d_1}{d_2}$ is certainly positive (the distances are signed!). This is so because, by assumption, S_1 and S_2 have no common tangents, and so either they lie one within the other, and thus on the same side of the radical axis o (Figure 110a), or they intersect and have the same direction, which means that p_1 and p_2 lie on the same side of the common chord o (Figure 110b).

Suppose that S_1 and S_2 are known to us. We apply an axial inversion whose central line is their radical axis o (whose direction can be chosen arbitrarily) and whose power is $\sqrt{\frac{d_1}{d_2}}$, where d_1 and d_2 are defined as above. Then the lines p_1 and p_2 go over to lines p_1' and p_2' that differ only by direction. The circles S_1 and S_2 go over to circles S_1' and S_2' which belong to the pencil of circles that share with S_1 and S_2 their radical axis o. But two different nondirected circles cannot be tangent to a line that is parallel to the radical axis of the pencil, that is, perpendicular to the line of centers (see

Figures 43b, 44b, and 46b which show pencils of nonintersecting, tangent, and intersecting circles). Hence S_1' and S_2' differ only by direction. This completes the proof of Theorem 2.

We note that the central line o and the directing circle Σ of an axial inversion that takes two given circles without common tangents to two given circles that differ only by direction can be constructed by ruler and compass. In fact, o is the radical axis of S_1 and S_2. As for Σ, the ratio of the distances l_1 and l_2 from the tangents b and b' to this circle—which are respectively parallel and counterparallel to o—to the central line o equals the power k of the inversion (see p. 107 and pp. 105–106). Therefore $\frac{l_1}{l_2} = \sqrt{\frac{d_1}{d_2}}$, that is, $\frac{l_1}{l_2}$ is equal to the ratio of the legs of the right triangle whose altitude divides its hypotenuse into segments d_1 and d_2. This condition makes possible the construction of Σ (Figure 111). The central line and directing circle of an axial inversion that takes two given circles to two points, or to a point and a circle passing through it, can also be constructed by ruler and compass (see pp. 111–113).

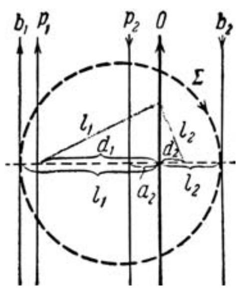


FIGURE 111

73. (a) Let S_1, S_2, S_3, and S_4 be four given circles such that each of S_1 and S_3 is tangent to S_2 and S_4 (Figure 112). Show that the four common tangents of the pairs of tangent circles pass through a point or are tangent to a circle.

[Here it is required that an even number (0, 2, or 4) of the four pairs of tangent circles—S_1 and S_2, S_2 and S_3, S_3 and S_4, S_4 and S_1—be inner-tangent and an even number of them be outer-tangent.[26]]

74. Construct a circle tangent to the following given figures:

(a) a line l and two circles S_1 and S_2 (or tangent to a given line l and a circle S_1 and passing through a given point S_2);

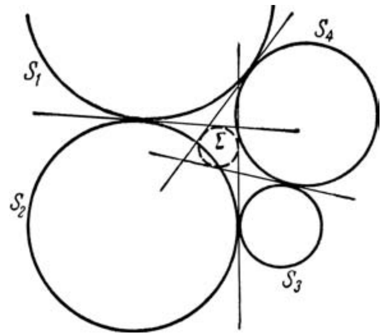


FIGURE 112

(b) three circles S_1, S_2, and S_3 (or tangent to two given circles S_1 and S_2 and passing through a given point S_3) (the problem of Apollonius).

See also problems 26(a) and 21(b) in Section 2.

75. Given three circles S_1, S_2, and S_3, of which S_1 is in the interior of S_2. Construct a circle Σ such that

(a) the tangent distance of S_1 and Σ is a, the tangent distance of S_2 and Σ is b, and the tangent distance of S_3 and Σ is c; a, b, and c are given;

(b) the angle between S_1 and Σ is α, the angle between S_2 and Σ is β, and the angle between S_3 and Σ is γ; $90^\circ > \alpha > \beta$.

See also problems 27(b) and 27(a) in Section 2 (p. 37).

76. Given four circles S_1, S_2, S_3, and S_4, with S_1 in S_2. Construct a circle Σ such that the tangent distances of the four pairs of circles S_1 and Σ, S_2 and Σ, S_3 and Σ, and S_4 and Σ are all the same.

To emphasize the duality between ordinary inversion and axial inversion we set down the definitions and basic properties of these transformations, as well as a selection of corresponding theorems—some proved and some new—in pairs.

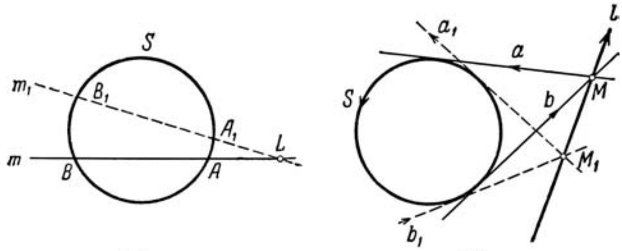
I. If m is a line passing through a fixed point L and intersecting a given circle S in two points A and B, then the product

$$LA \cdot LB$$

(Figure 113a) depends only on L and S and not on m (Theoren $1'$, p. 101). This product is called the power of L with respect to S.

I′. If M is a point on a fixed (directed) line l outside a given (directed) circle S and a and b are tangents from M to S, then the product

$$\tan \frac{\widehat{la}}{2} \cdot \tan \frac{\widehat{lb}}{2}$$

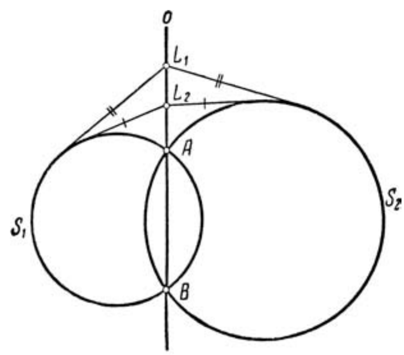


(a) (b)

FIGURE 113

(Figure 113b) depends only on l and S and not on M (Theoren 1, p. 100). This product is called the power of l with respect to S.

II. The locus of points L whose powers with respect to two given circles S_1 and S_2 are equal is a line o (see Section 3). This line is called the radical axis of the two circles. If S_1 and S_2 have common points A and B, then these points are on the radical axis (Figure 114a). The radical axis of two circles can be defined as the line such that the tangent distances from any of its points to S_1 and S_2 are equal.

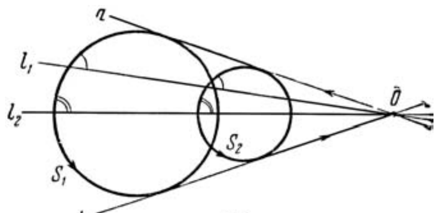


(a)

FIGURE 114

II′. All (directed) lines l whose powers with respect to two (directed) circles S_1 and S_2 are equal pass through a point O (the reader should try to prove this). This point is called the center of similarity of the two circles. If S_1 and S_2 have common tangents a and b, then these tangents pass through the center of similarity (Figure 114b). The center of similarity of two circles can be defined as a point such that all lines through it form equal angles with both circles.

III. The locus of points such that the ratio of their powers with respect to two circles S_1 and S_2 has a constant value other than 1 is a circle (see problem 39(b) in Section 3).[27]

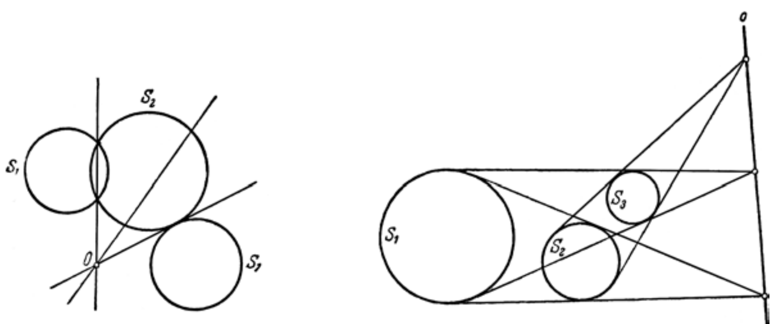


(b)

FIGURE 114

III′. All (directed) lines the ratio of whose powers with respect to two directed circles S_1 and S_2 has a constant value other than 1 are tangent to some circle (the reader should try to prove this).[28]

IV. The radical axes of circles S_1, S_2, and S_3 taken two at a time intersect in a point O (see, for example, Figure 115a). This point is called the radical center of the three circles (see p. 53).



(a) (b)

FIGURE 115

IV′. The centers of similarity of circles S_1, S_2, and S_3 taken two at a time lie on a line (see, for example, Figure 115b). This line is called the axis of similarity of the three circles (see p. 29 in NML 21).[29]

V. An inversion with center O and power k is a transformation that takes a point A to a point A' such that A, A', and O lie on a line and

$$OA \cdot OA' = k.$$

V′. An axial inversion with central line o and power k is a transformation that takes a line a to a line a' such that a, a', and o intersect in a point and

$$\tan \frac{\widehat{oa}}{2} \cdot \tan \frac{\widehat{oa'}}{2} = k.$$

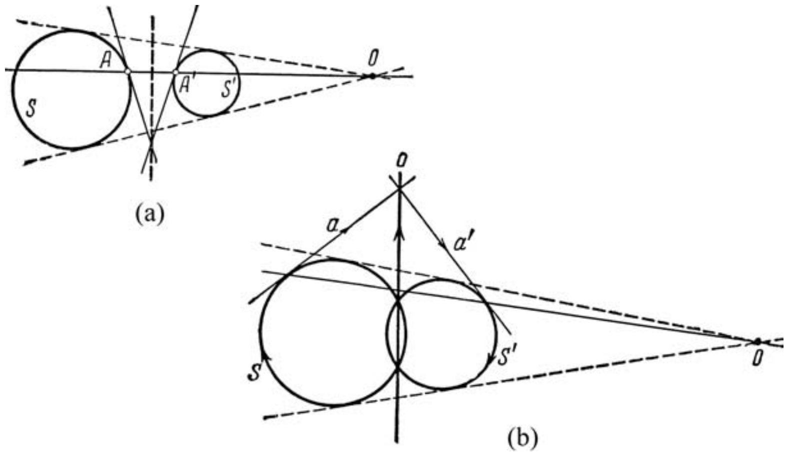


FIGURE 116

VI. An inversion takes a circle S to a circle S' (see property B of inversion, p. 10). Here "circles" includes lines ("circles of infinite radius").

VI′. An axial inversion takes a circle S to a circle S' (see property B of axial inversion, p. 108). Here "circles" includes points ("circles of zero radius").

VII. The center O of an inversion that takes a circle S to a circle S' is the center of similarity of S and S' (Figure 116a; see the proof of property B of inversion, pp. 10–11).

VII′. The central line o of an axial inversion that takes a circle S to a circle S' is the radical axis of S and S' (Figure 116b; see the proof of property B of axial inversion, pp. 108–111).

VIII. Consider an inversion that takes a circle S to a circle S'. The locus of the points of intersection of the tangents to S and S' at points A and A' that correspond under the inversion is a line—the radical axis o of S and S' (Figure 116a; see problem 40(a) in Section 3).

VIII′. Consider an axial inversion that takes a circle S to a circle S'. The lines joining the points of tangency of tangents a and a' to S and S' at points A and A' that correspond under the axial inversion pass through a fixed point—the center of similarity of S and S' (Figure 116b; see the proof of property B of axial inversion).

IX. An inversion preserves the angle between two circles (property C of inversion, p. 11).

IX′. An axial inversion preserves the tangent distance between two circles (property C of axial inversion, p. 116).

X. An inversion preserves the cross ratio

$$\frac{AC}{BC} : \frac{AD}{BD}$$

of four points A, B, C, and D (property D of inversion, p. 63).

X′. An axial inversion preserves the cross ratio

$$\frac{\sin\frac{\widehat{ac}}{2}}{\sin\frac{\widehat{bc}}{2}} : \frac{\sin\frac{\widehat{ad}}{2}}{\sin\frac{\widehat{bd}}{2}}$$

of four lines a, b, c, and d (see problem 70, p. 120).

XI. The totality of circles any two of which have the same radical axis o is called a *pencil of circles* (see p. 56). The line o is called the axis of the pencil. If two circles of the pencil intersect at A and B, then all circles of the pencil pass through A and B (Figure 117a).

XI′. The totality of circles any two of which have the same center of similarity O is called a row of circles. The point O is called the center of the row. If two circles of the row have common tangents a and b, then all circles of the row are tangent to a and b (Figure 117b).

XII. The totality of circles with respect to which a given point O has the same power is called a bundle of circles. The point O is called the center of the bundle. If the center O lies outside a circle of the bundle, then the tangent distance between O and each of the circles of the bundle has the same value.

XII′. The totality of circles with respect to which a given line o has the same power is called a net of circles. The line o is called the central line of the net. If the circles of the net intersect the central line o, then all of them form equal angles with it.

XIII. The circles common to two different bundles form a pencil.

XIII′. The circles common to two different nets form a row.

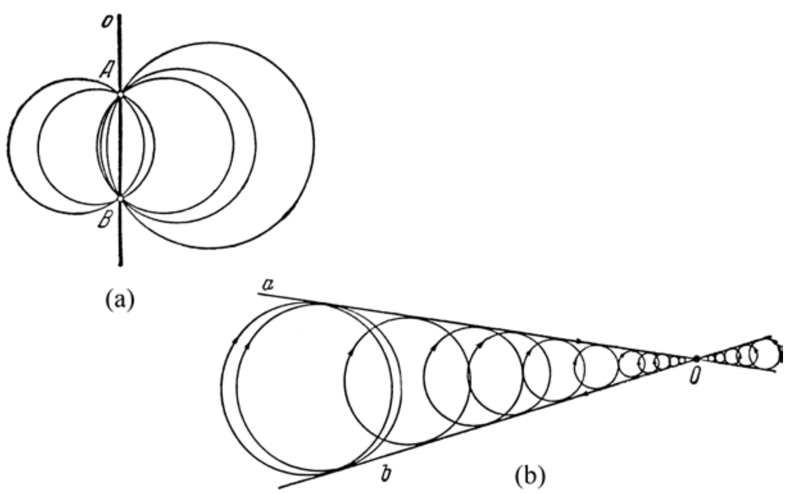


FIGURE 117

XIV. An inversion takes a pencil of circles to a new pencil and a bundle of circles to a new bundle.

XIV′. An axial inversion takes a row of circles to a new row and a net of circles to a new net.

XV. The totality of circles that intersect two given circles at the same angles forms a bundle of circles whose center is the center of similarity of the two given circles.[30]

XV′. The totality of circles whose tangent distances from two given circles are equal forms a net of circles whose central line is the radical axis of the two given circles.

XVI. The totality of circles that intersect three given circles at the same angles forms a pencil of circles whose axis is the axis of similarity of the three given circles. The centers of all circles of this pencil lie on the perpendicular dropped from the radical center of the three circles to their axis of similarity.[30]

XVI′. The totality of circles whose tangent distances from three given circles are equal forms a row of circles whose center is the radical center of the three given circles. The centers of all circles of this row lie on the perpendicular dropped from the radical center of the three circles to their axis of similarity.

This collection of theorems could be greatly enlarged.

Finally, we note two theorems analogous to Theorems 1 and 2 in Section 4:

Theorem 3. *Every circular transformation which is a point-to-point transformation is a similarity*.[31]

Theorem 4. *Every axial circular transformation which is not a similarity can be realized by a dilatation or an axial inversion, possibly followed by a similarity.*[31]

It is obvious that Theorem 3 is equivalent to Theorem 1 in Section 4. Specifically, both theorems express the fact that every transformation of the plane that takes points to points, lines to lines, and circles to circles is a similarity.

As for Theorem 4, its proof—which is conceptually close to that of Theorem 2 in Section 4—is fairly complicated. We provide a sketch of this proof.

The key to the theory of circular transformations of the plane is stereographic projection. For example, it is convenient to regard inversion as a transformation of the plane π corresponding to a reflection of the sphere σ to which the plane is mapped by stereographic projection (see Section 4, pp. 74–77); from this one can easily deduce all properties of inversion (the reader should try to do this). All definitions of inversion which do not stress its connection with reflection in space (the first definition on p. 5 and the second definition on p. 5) are basically more artificial, and do not explain why this transformation plays such an important role in the theory of circular transformations. Stereographic projection is also the

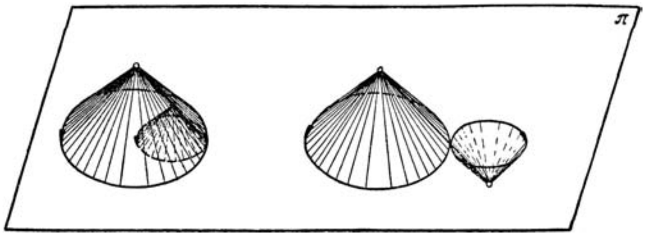


FIGURE 118

key to the theory of cyclographic projections of Chasles-Fedorov, dual to circular transformations. The Chasles-Fedorov transformations are defined as follows.

With every (directed) circle in the plane π we associate a point in space, namely, the vertex of a right-angled cone, that is, a cone whose angle at the vertex is 90° and whose base is the given circle (Figure 118). Depending on the clockwise or counterclockwise direction of the circle, the cone is located above or below the plane π. Thus a cyclographic projection maps circles in the plane to points in three-dimensional space. As for points in the plane, their images under a cyclographic projection are these very points.

It is clear that the images of tangent circles are tangent cones (Figure 118). This implies that the line l joining the (point) images of two tangent circles forms a 45° angle with π. The images of circles tangent to the line l are points in the plane λ

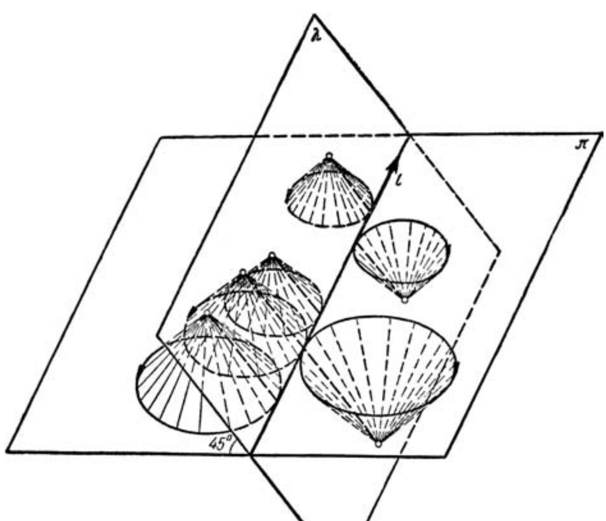


FIGURE 119

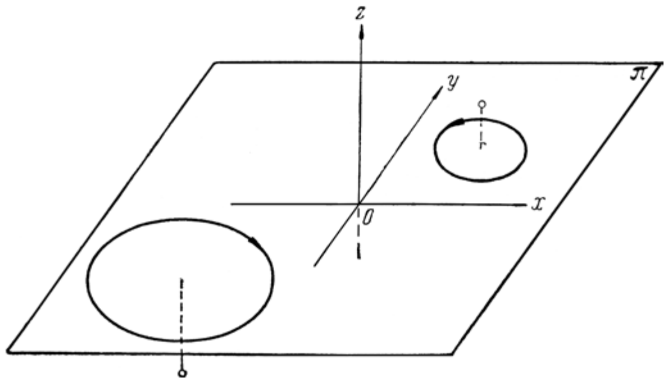


FIGURE 120

passing through l and inclined to π at 45° (Figure 119).[32] Therefore it is natural to view the plane λ as the image of the line l under a cyclographic projection.[33]

It is convenient to view the plane π as the coordinate plane xOy (Figure 120). Then we can view a cyclographic projection as follows: to a circle in π whose center has coordinates x, y and whose radius (positive or negative) is r there corresponds in space the point with coordinates x, y, and $z = r$. To two tangent circles there correspond two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in space such that the segment joining them forms a 45° angle with π; then the projection $|z_1 - z_2|$ of the segment on the z-axis is equal to its projection

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \text{on} \quad \pi$$

(Figure 121). Hence

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (z_1 - z_2)^2,$$

or

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2 = 0.$$

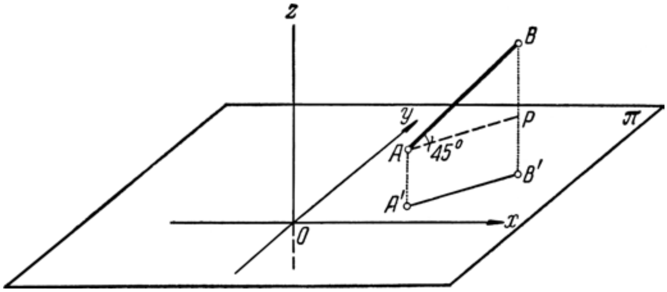


FIGURE 121

Axial circular transformations take circles in the plane to circles. Hence a cyclographic projection associates with such a transformation a point-to-point transformation of space. Also, these transformations take lines to lines. This implies that *tangent circles* (with a unique common tangent) *must go over to tangent circles*. But then a cyclographic projection associates with an axial circular transformation a transformation of space such that two points whose coordinates satisfy the relation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2 = 0$$

go over to two points satisfying this very relation.

It is possible to show that a transformation of space with the property just described can be represented as the product of transformations that preserve the magnitude of the expression

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2 \tag{$*$}$$

and a central similarity (dilatation) of space with center at the origin[34] which changes ($*$) in some definite way (in particular, this result is implied by our subsequent considerations). But it is easy to see that a cyclographic projection associates with a central similarity of space with center at the origin a central similarity of the plane π. Hence the study of axial circular transformations reduces to the study of *transformations of space that preserve the expression* ($*$), that is, transformations which take any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in space to two points (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2) such that[35]

$$(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 - (z'_1 - z'_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2.$$

Transformations that preserve ($*$) have much in common with isometries of space, that is, transformations that preserve the expression

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{$**$}$$

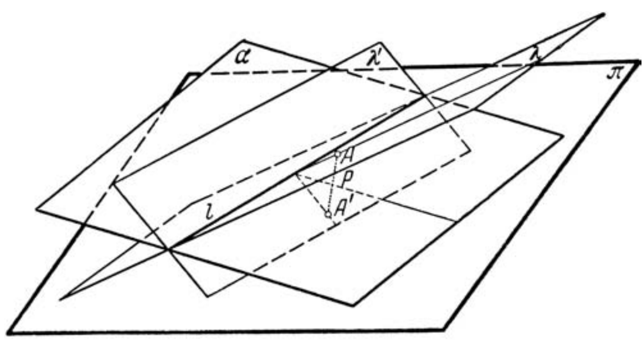
(see the footnote on p. 11 in NML 8). We will call them pseudoisometries. One might call the study of the properties of figures in space preserved under pseudoisometries of space pseudogeometry (or—to use the more popular term—pseudo-Euclidean geometry). Pseudogeometry plays a very important role in modern physics (more specifically, in the theory of relativity). This geometry has much in common with ordinary school geometry. The pseudogeometric analogue of the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in space is the expression

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2}$$

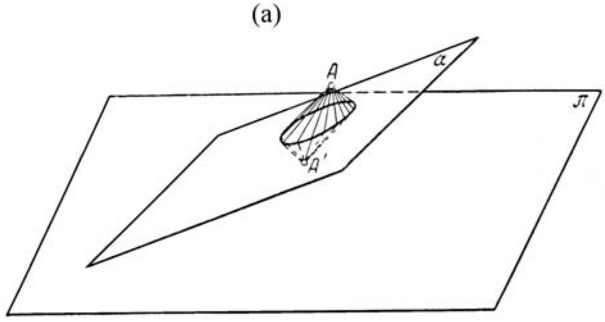
which can take on complex values. Analogously to Euclidean geometry, two segments AC and BC are said to be pseudoperpendicular if the theorem of Pythagoras holds for the triangle ABC, that is, if the square of the pseudodistance AB is equal to the sum of the squares of the pseudodistances AC and BC.[36] Also, from every point one can drop a pseudoperpendicular to every plane (that is, a line pseudoperpendicular to all lines in the plane), and two pseudoperpendiculars to the same plane are parallel (in the ordinary sense of the term: parallelism of lines and planes in pseudogeometry has the same meaning as in Euclidean geometry), and so on.[37]

We will be concerned with the notion of pseudoreflection in a plane. We will say of two points A and A' in space that they are pseudoreflections of one another with respect to a plane α if the segment AA' is pseudoperpendicular to α and is halved by it (in the sense that the pseudodistances AP and AP', where P is the point of intersection of AA' and α, are equal; incidentally, pseudoequal segments are equal in the ordinary sense of the (latter) term). It is easy to show that, under a pseudoreflection in α, *a plane* λ *that forms with the initial plane* π *a* $45°$ *angle goes over to a plane* λ' *that also forms with* π *a* $45°$ *angle and intersects* α *in the same line* l *as* λ (Figure 122a;[38] this can be taken as a definition of a pseudoreflection in α, for it enables us to determine the image A' of a point A in space under this transformation (Figure 122b; here the two cones with vertices A and A' are such that all planes tangent to these cones form a $45°$ angle with π; in other words, the generators of these cones form $45°$ angles with π). It is easy to see that pseudoreflections are a special case of pseudoisometries, that is, that they preserve the pseudodistance between points.[39] One can also show that two planes which can be taken to one another by a pseudoisometry can be taken to one another by a pseudoreflection in a suitably chosen plane.[40]

After all these preliminaries we can go over to the proof of Theorem 4. A cyclographic projection associates with every axial circular transformation Λ which takes



(a)



(b)

FIGURE 122

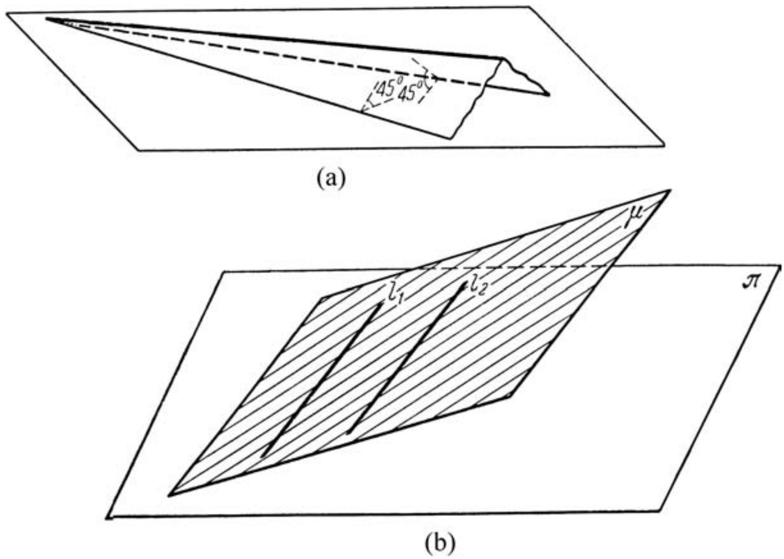


FIGURE 123

circles in π to circles a point-to-point transformation $\overline{\Lambda}$ that takes every plane λ which forms a 45° angle with π to another such plane (in virtue of the cyclographic projection these planes correspond to the lines in π). We now show that $\overline{\Lambda}$ *takes every plane in space to another plane* (and is therefore an affine transformation of space). Through every line in space that forms with π an angle smaller than 45° one can pass two planes that form with π a 45° angle (Figure 123a). It follows that $\overline{\Lambda}$ takes such a line to another such line (two planes that form 45° angles with π go over to two new planes, and their line of intersection goes over to the line of intersection of their image planes). Let μ be an arbitrary plane. We choose in μ two lines l_1 and l_2 that form with π angles smaller than 45° (we can find arbitrarily many such lines in every plane; we can choose such lines that form with π "zero" angles, that is, lines parallel to π) and consider all lines that intersect l_1 and l_2 and form with π angles smaller than 45° (the lines shown in Figure 123b are parallel to one another and intersect l_1 and l_2—it suffices to consider just such lines). The transformation $\overline{\Lambda}$ takes l_1 and l_2 to new lines l_1' and l_2', and lines intersecting l_1 and l_2 to lines intersecting l_1' and l_2'; this implies that $\overline{\Lambda}$ takes the plane μ covered by the former lines to a new plane μ' (determined by l_1' and l_2').

We note that *if* $\overline{\Lambda}$ *takes* π *to itself, then there corresponds to it a similarity of* π. In fact, the cyclographic projection associates with points of space in π points ("circles of zero radius") in π; it follows that if $\overline{\Lambda}$ takes points in π to points in π, then the axial circular transformation Λ must take points to points, that is, it must be a similarity (see Theorem 3).

Now suppose that $\overline{\Lambda}$ does not fix π, and let π_1 be the plane that $\overline{\Lambda}$ takes to π. We consider two cases.

1°. The plane π_1 is parallel to π. Let $\overline{P}$ be the translation of space in the direction of the z-axis (perpendicular to π and π_1) that takes π_1 to π;[41] $\overline{P}$ is a pseudoisometry that takes π_1 to π. Further, let $\overline{\Lambda}_1$ be a transformation of space such that $\overline{\Lambda}$ is representable as a product of $\overline{P}$ and $\overline{\Lambda}_1$ (see pp. 73–77). Clearly, $\overline{\Lambda}_1$ fixes π (for $\overline{\Lambda}$ and $\overline{P}$ take π_1 to π). It follows that the axial circular transformation Λ_1 of π that corresponds to the transformation $\overline{\Lambda}_1$ of space is a similarity. Clearly, to the transformation $\overline{P}$ of space there corresponds a dilatation P of π. In fact, $\overline{P}$ takes a point (x, y, z) in space to a point $(x, y, z + a)$, where a is the distance between π and π_1. It follows that P takes the circle with center (x, y) and (positive or negative) radius r to the circle with the same center and radius $r + a$. Since $\overline{\Lambda}$ is the product of $\overline{P}$ and $\overline{\Lambda}_1$, it follows that the initial axial circular transformation Λ is the product of the dilatation P and the similarity Λ_1.

2°. The plane π_1 is not parallel to π. Let $\overline{\Omega}$ be the pseudoreflection that takes π_1 to π,[42] and $\overline{\Lambda}_1$ a transformation such that $\overline{\Lambda}$ is the product of $\overline{\Omega}$ and $\overline{\Lambda}_1$. Clearly, $\overline{\Lambda}_1$ fixes π. Hence to $\overline{\Lambda}_1$ there corresponds a similarity Λ_1 of the plane. We show that to the pseudoreflection $\overline{\Omega}$ there corresponds an axial inversion Ω. This will complete the proof of Theorem 4.

Let o be the line of intersection of π_1 and π. Clearly, α also passes through o. Hence $\overline{\Omega}$ (and the axial circular transformation Ω that corresponds to it) fixes every point on o. Further, let $\overline{S}$ be a point in the plane α and let S be the circle whose image under the cyclographic projection is $\overline{S}$ (Figure 124). The transformation $\overline{\Omega}$ takes every plane λ passing through $\overline{S}$ and inclined to π at 45° to a plane λ' that is likewise inclined to π at 45° and intersects α in the same line as λ (see Figure 122a). It follows that λ' also passes through $\overline{S}$ and intersects o at the same point M as λ. This implies that Ω takes every line l tangent to the circle S to a line l' likewise tangent to S and intersecting o at the same point M as λ (Figure 124). Since, in addition, Ω takes parallel lines to parallel lines (for $\overline{\Omega}$ takes planes inclined to π at 45° and parallel to one another—that is, planes that intersect α along parallel lines—to parallel planes), it follows that Ω is an *axial inversion* with central line o and directing circle S (see the definition on p. 105).

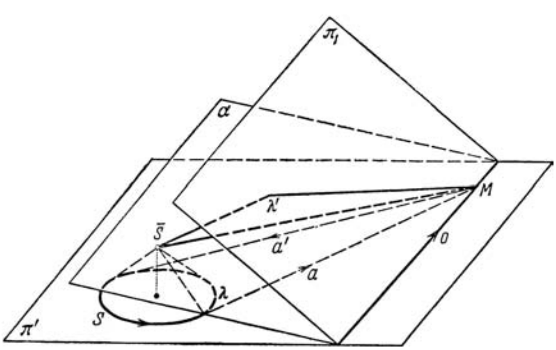


FIGURE 124

Notes to Section 5

[1] Some of the circles S_1, S_2, S_3, and S_4 can be points.

[2] The bisector of the angle formed by two directed lines can be defined as the (directed) line that forms equal angles with both sides of the angle, or as the locus of points whose distances from the sides of the angle are equal (in magnitude and sign). These two definitions do not coincide. Here we use the term "angle bisector" in the sense of the first definition, whereas in Theorem V′ (p. 96) we use it in the sense of the second definition.

[3] Here is another important example of the same kind: *if one can draw two* (directed) *tangents m and n to a directed circle S, and if M and N are points of S on these tangents, then the segments AM and AN have the same magnitudes and opposite directions* (Figure 86a). Similarly, *if a line a intersects a circle S in points M and N and m and n are tangent to S at these points, then the angles between a and m and a and n are equal in magnitude and opposite in direction* (Figure 86b).

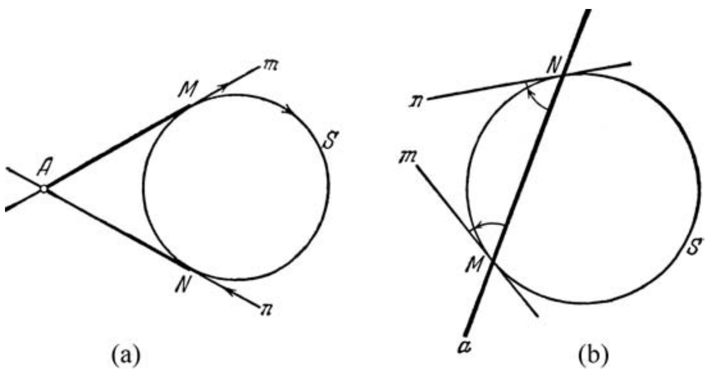


FIGURE 86

[4] Using the earlier analogy between the basic properties of points and directed lines one can assert without proof the validity of one of two corresponding propositions from the validity of the other (see the discussion of the "principle of duality" in the projective plane on pp. 77–78 in NML 24). We will not discuss this issue in greater detail.

[5] Here we again use the analogy between distances and angles encountered in Note 4.

[6] The reasons why it is convenient to assume that the principle of duality associates directed lines with points are discussed in the text in fine print on p. 94f.

[7] We note that if one views a dilatation as an axial transformation, then there is no need to state in the definition of a dilatation to what figures it takes circles (an axial transformation is fully determined by stating to what line this transformation takes an arbitrary line). From our new viewpoint, the fact that a dilatation takes a circle to a circle implies that a dilatation takes all lines tangent to a circle S to tangents of its image circle S'. This fact is a consequence of the rule which defines the effect of a dilatation on lines.

[8] In the literature these transformations are usually referred to as Laguerre transformations. (Laguerre was a famous French mathematician who first investigated these transformations.)

[9] To avoid misunderstandings it is vital to stress that the content of this section suggests the principle of duality but that this principle is not deducible from it. Hence an argument such as, say, the following: "point-to-point circular transformations preserve the angle between circles and therefore axial circular transformations should preserve the tangent distance between two circles, a concept that corresponds to the concept of an angle under the principle of duality" is, of course, completely unfounded. We know certain properties of polarities and we based the principle of duality on them (see Section 4 in NML 24), but these properties provide no basis for such a conclusion. In order to use the principle of duality to deduce the properties of axial circular transformations from the properties of point-to-point circular transformations it would be necessary to develop this principle far more extensively than we have done in this book; in fact, for this we would have to write a book the size of this book.

[10] We will denote the angle between directed lines a and b by $\widehat{ab}$.

[11] It is easy to verify that $(*)$ holds for other choices of M on l (see the dotted lines in Figures 90a and 90b).

If the circle S has a negative direction, or O lies to the right of l, or both of these conditions hold at the same time, then the value of the product $\tan\frac{\widehat{la}}{2}\cdot\tan\frac{\widehat{lb}}{2}$ will also equal $\left|\frac{r-d}{r+d}\right|$, where r is the radius of the directed circle S (and can be positive or negative; see p. 86) and d is the distance from O to the directed line l (which can also be positive or negative; see p. 88). The reader should investigate all possible cases.

[12] It is well known that *if a point L lies outside a circle S, then its power with respect to S is equal to the square of the tangent distance between L and S.* Similarly, *if a* directed *line l intersects a* directed *circle S, then its power with respect to S is equal to the square of the tangent of half the angle between l and S*; for proof it suffices to take as the point M in Figure 90a the point of intersection of l and S.

[13] The product $LA \cdot LB$ (the power of L with respect to S) is $|D^2 - r^2|$, where r is the radius of S and D is the distance between L and the center O of S. Hence the product

$$\tan\frac{\widehat{la}}{2}\cdot\tan\frac{\widehat{lb}}{2} \quad \text{is equal to} \quad \frac{D^2-r^2}{(D+r)^2} = \left|\frac{D-r}{D+r}\right|.$$

Since $D = \frac{r^2}{d}$, where d is the distance of l from O (see problem 51 on p. 70 of NML 24), $|\frac{D-r}{D+r}| = |\frac{r-d}{r+d}|$ (see p. 100).

[14] In other words, at the point of intersection of a, a', and o, the arrows on a and a' are directed to one side of o.

[15] That is, parallel to o and oppositely directed.

[16] If k_0 and l_0 are parallel, then o passes through the center of Σ. In that case the axial inversion reduces to a reflection in o with a subsequent change of the directions of all lines, and property B is obvious.

[17] The reader should try to modify the proof so that it becomes applicable to an axial inversion with negative power.

[18] In general, three nondirected circles have four axes of similarity. Three directed circles have a unique axis of similarity (see p. 94, in particular, Figure 85).

[19] The difference between an "ordinary" inversion and an axial inversion comes to the fore here. The case when an inversion takes three circles to three lines is a rare exception (for this to occur the three circles must be concurrent). In contradistinction to this, it is very often possible to take three circles to three points by an axial inversion (the case when the axis of similarity does not intersect these circles cannot be viewed as an exception; roughly speaking, this case occurs "just as often" as the opposite case).

[20] To make this argument independent of figures we must use the notion of directed line segments (compare this with the first footnote on p. 80 in NML 8).

[21] In problems 67–69 common tangents of circles must be chosen so that one can view them as directed segments of directed circles (regardless of the choice of direction on the circles).

For example, in problem 67(a) we must require a_1 and a_2 to be "same-name" common tangents of S_1 and S_2 (both must be inner or both outer). Similarly, b_1 and b_2, c_1 and c_2, and d_1 and d_2 must be "same-name" tangents. Also, one must see to it that among the four pairs of common tangents—a_1, a_2; b_1, b_2; c_1, c_2; and d_1, d_2—the number of pairs of inner tangents is even: zero (see Figure 107a), two, or four. But even if both of these conditions are satisfied it is not certain that the problem is well posed. A third requirement is that the circles S_1, S_2, S_3, and S_4 are so directed that their directed tangents a_1, b_1, c_1, and d_1 are tangent to a single directed circle Σ. (It is clear that such a choice of directions on the circles S_1, S_2, S_3, and S_4 in Figure 108 is impossible. In this figure a_2, b_2, c_2, and d_2 are not tangent to any circle.)

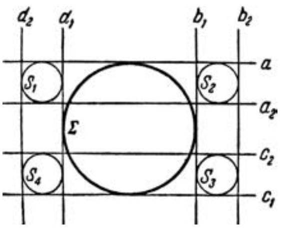


FIGURE 108

[22] See Note 21.

[23] There is an obvious analogy between the content of this problem and the properties of the medians of a triangle. In this connection see p. 161.

[24] See Note 15.

[25] Here there is a minus sign on the right side. This is so because, if k is positive, then the initial line and the transformed line lie on different sides of o (see pp. 105–106).

[26] This condition secures the possibility of viewing the circles S_1, S_2, S_3, and S_4 as directed, and takes into consideration the conditions of tangency of directed circles.

[27] This circle belongs to the pencil of circles determined by S_1 and S_2 (see proposition XI on p. 127).

[28] This circle belongs to the row of circles determined by S_1 and S_2 (see proposition XI′ on p. 127).

[29] This proposition can be proved in a manner entirely analogous to the proof of the theorem on radical axes (see p. 53). From this we obtain a new proof of the theorem on three centers of similarity (if the figures F, F_1, and F' considered in this theorem are not circles, then to carry out the proof it suffices to circumscribe about three corresponding points A, A_1, and A' of these figures circles such that the ratios of their radii are equal to the coefficients of similarity of these figures).

[30] Here we must consider directed circles; the angle between directed circles is uniquely determined as the angle between their directed tangents at a point of intersection of these circles. The totality of circles that intersect two nondirected circles at equal angles consists of two bundles with centers at the two centers of similarity of the circles; the totality of circles that intersect three nondirected circles at equal angles consists of four pencils whose axes are the four axes of similarity of the circles.

[31] Here, and in all questions involving directed circles and lines, by a similarity we mean a direct or opposite similarity (see Section 2 in NML 21), possibly followed by a reversal of the directions of all lines and circles.

[32] To bring λ into coincidence with π we must rotate it counterclockwise through 45°—we are looking in the direction determined by the directed line l in Figure 119. This condition determines one of the two planes that intersect π along l and form with π a 45° angle.

[33] The reader may at first be baffled by the fact that, in questions dual to one another, analogous roles are played by stereographic projection and cyclographic projection, two seemingly radically different transformations: the first takes points in a plane to points on a sphere and the second takes circles in a plane to points in space. This is easy to clarify. Stereographic projection can be viewed as a transformation that takes circles in a plane to circles on a sphere, or—and this is the same thing—as *a transformation that takes circles in a plane to planes in space* containing those circles. *Stereographic projection takes points in a plane to certain points in space*, namely, points on the sphere σ. The duality principle in space sets up a correspondence between points and planes (which is analogous to the fact that in a plane it sets up a correspondence between points and lines). It is therefore natural that in the dual theory the role of stereographic projection is played by a projection that *takes circles in a plane to points in space,*

and lines in a plane to certain planes in space, that is, by cyclographic projection.

[34] See the second footnote on p. 30 in NML 21.

[35] A cyclographic projection associates with such transformations *axial circular transformations, which preserve the tangent distance between circles.* Indeed, if

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2 = d^2 > 0,$$

then d is the tangent distance between circles in the plane π to which there correspond the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in space (see the formula $(*)$ on p. 90).

[36] If OA and OB are two perpendicular segments in space—in the ordinary sense of the term—and (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of A and B, then the theorem of Pythagoras,

$$OA^2 + OB^2 = AB^2,$$

implies that

$$(x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

or

$$x_1x_2 + y_1y_2 + z_1z_2 = 0;$$

the latter equality can also be used as a condition of perpendicularity. Analogously, the pseudogeometric version of the theorem of Pythagoras implies that two segments OA and OB are pseudoperpendicular if and only if

$$x_1x_2 + y_1y_2 - z_1z_2 = 0$$

(here (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of A and B and O is the origin); this equality can be taken as a definition of pseudoperpendicularity of the segments OA and OB.

[37] The equation of a plane α in space can be written as

$$Ax + By + Cz = D.$$

From what was said in Note 36, it is easy to deduce that all lines symmetric with respect to Π to an ordinary perpendicular to α, that is, lines parallel to the segment OA, where O is the origin and A has coordinates $(A, B, -C)$—and only these lines—are pseudoperpendicular to α; all our assertions follow from this.

[38] In other words, any line m, pseudoperpendicular to α (see, for example, Note 37), intersects λ, α, and λ' in points A, P, and P' such that $AP = AP'$ (here the equality of segments can be viewed as equality in the ordinary sense or in the sense of pseudogeometry).

[39] Compare this with the proof of the fact that a reflection of a plane in a line is an isometry, that is, it does not change the length of segments (see Section 1, Chapter II of NML 8).

[40] Obviously, a plane μ that forms with π an angle smaller than 45° does not contain lines inclined to π at 45° (lines such that the pseudodistance of any two of their points is zero), and a plane ν that forms with π an angle not smaller than 45° does contain such lines. Hence no pseudoisometry can take μ to ν.

Further, we noted already that any two planes that form 45° angles with π are pseudosymmetric with respect to any plane passing through their line of intersection. Now let μ_1 and μ_2 be two planes forming with π angles smaller than (or larger than) 45°. Let OM_1 and OM_2 be segments of equal length pseudoperpendicular to μ_1 and μ_2, that is, such that $A_1^2 + B_1^2 - C_1^2 = A_2^2 + B_2^2 - C_2^2$, where $(A_1, B_1, -C_1)$ and $(A_2, B_2, -C_2)$ are the coordinates of M_1 and M_2 (see Note 37; if μ_1 formed with π an angle greater than 45° and μ_2 an angle smaller than 45°, than the pseudolength of OM_1 would be real and that of OM_2 would be imaginary). In that case μ_1 and μ_2 are pseudosymmetric with respect to the plane α passing through the line of intersection of μ_1 and μ_2 and pseudoperpendicular to the "vector sum" OA of OM_1 and OM_2, that is, the segment joining O to $A(A_1 + A_2, B_1 + B_2, -C_1 - C_2)$.

[41] The properties of translations in space are analogous to properties of translations in the plane.

[42] The plane π_1 must form with π an angle smaller than 45°. Otherwise π_1 would contain points that lie on a line inclined to π at 45° that $\overline{\Lambda}$ cannot take to points in π (this is so because the axial circular transformation Λ cannot take tangent circles to points). It follows that π can be obtained from π_1 by a pseudoreflection in some plane α (see Note 40).

Supplement

Non-Euclidean Geometry of Lobachevski-Bolyai, or Hyperbolic Geometry

Second account

(The first account is found on pp. 103–135 of NML 24 and is known as the Klein model of hyperbolic geometry. The present account is known as the Poincaré model of hyperbolic geometry.)

Let $\mathbb{K}$ be a disk in the plane. We consider all circular transformations of the plane (see pp. 72–74) that take $\mathbb{K}$ to itself. We call the points of $\mathbb{K}$ the points of hyperbolic geometry and these circular transformations—non-Euclidean motions. We will call the geometry concerned with the properties of figures preserved by non-Euclidean motions hyperbolic geometry (cf. NML 24, p. 104).[1]

It is easy to see that given any point A of hyperbolic geometry (that is, any interior point of $\mathbb{K}$) there is a non-Euclidean motion which takes it to any other such point. For example, in order to take A to the center O of $\mathbb{K}$ we erect at A the perpendicular to OA and denote by M and N the points in which it intersects Σ, the boundary of $\mathbb{K}$. If R is the point of intersection of the tangents to Σ at M and N (Figure 125), then, in view of the similarity of the triangles ROM and RMA, we have $\frac{RO}{RM} = \frac{RM}{RA}$, or $RO \cdot RA = RM^2$. Hence the inversion with center R and power RM^2 (which takes $\mathbb{K}$ to itself and is therefore a non-Euclidean motion) takes A to O (and O to A). If we wish to take A to a point A' different from O, then we need only take A to O and O to A' (by the very same non-Euclidean motion which takes A' to O).

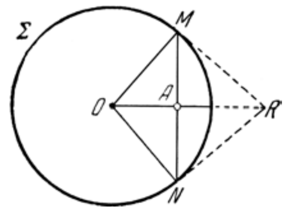


FIGURE 125

Now we define non-Euclidean lines. It is obvious that if we expect non-Euclidean motions to take lines to lines, then we must not call ordinary lines non-Euclidean lines—after all, as a rule, circular transformations take lines to circles. Suppose we defined as non-Euclidean lines all arcs of circles (and segments of lines) intersecting $\mathbb{K}$. Then motions would take "lines" to "lines" but, given any two points, there would be a great many "lines" passing through these points, and this would go against our expectations. This being so, it is natural to call "lines" only some of the circles and lines intersecting $\mathbb{K}$. Likewise, it is natural to require that a non-Euclidean motion should take the totality of these circles and lines to itself, and that just one "line" should go through any two interior points of $\mathbb{K}$. Both of these requirements are satisfied by the circles (and lines) perpendicular to the circle Σ. In fact, any non-Euclidean motion takes such a circle or line to another such circle or line (see property C of inversion on p. 11 and the remark about circular transformations on p. 72). On the other hand, all circles (and lines) perpendicular to Σ and passing through a definite point A pass through the point $\overline{A}$ symmetric to A with respect to Σ (compare Figure 126 with Figure 6 on p. 4), that is, they form a pencil of intersecting circles. But there is exactly one circle of such a pencil that passes through a point

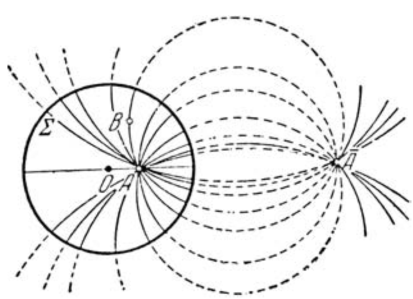


FIGURE 126

B other than A and $\overline{A}$ (there is exactly one circle or line passing through three points A, B, and $\overline{A}$; see also Section 3, p. 45). Thus in hyperbolic geometry there is exactly one circle passing through any two points A and B and perpendicular to Σ. Also, Figure 126 makes it clear that in any given direction at A there is exactly one circle perpendicular to Σ that passes through A in that direction. We therefore agree to call arcs of circles (and segments of lines[2]) perpendicular to Σ and contained in the interior of $\mathbb{K}$ lines of hyperbolic geometry; in particular, the non-Euclidean lines through the center O of $\mathbb{K}$ are the diameters of $\mathbb{K}$.

In hyperbolic geometry we define a ray issuing from a point A as an arc perpendicular to Σ, passing through A, and bounded by A and Σ. It is easy to see that given a ray AP and a ray $A'P'$ there is a non-Euclidean motion which takes AP to $A'P'$. In fact, let E_1 be a motion that takes A to the center O of $\mathbb{K}$ and the ray AP to some ray OQ (Figure 127), and let E_2 be a motion that takes O to A' and the ray OQ' to the ray $A'P'$ (see p. 143). Then the product of the three motions: E_1, the rotation C about O through the angle $Q'OQ$, and E_2, take the ray AP to the ray $A'P'$ (see pp. 104–105 in NML 24).

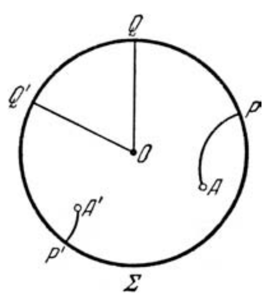
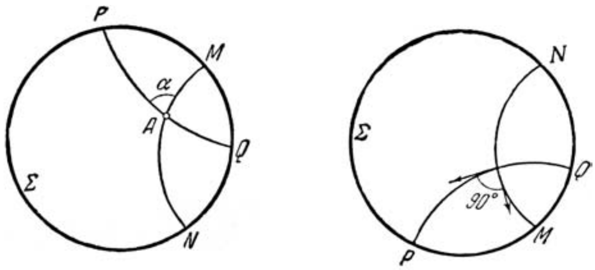


FIGURE 127

Next we define the terms *distance* between two points and *angle* between two lines in hyperbolic geometry. The latter is the simpler one of the two terms. Since circular transformations preserve angles between circles (see property C of inversion on p. 11 and the remark on p. 72), *we can define the non-Euclidean angle between two hyperbolic lines PQ and MN which intersect in a point A to be the ordinary* (Euclidean) *angle between the circles PAQ and NAM* (Figure 128a). It follows that in hyperbolic geometry, just as in ordinary (Euclidean) geometry, all full angles about a point are equal, and so too are all straight angles; specifically, a full



(a) (b)

FIGURE 128

angle is 360° and a straight angle is 180°. Also, a right angle (equal to its adjacent angle) is 90°, and so on. Two non-Euclidean lines PQ and MN are perpendicular in the sense of non-Euclidean geometry (that is, form a right angle) if and only if the circles PQ and MN are perpendicular in the ordinary sense of the term (Figure 128b). From the viewpoint of Euclidean geometry, the totality of all non-Euclidean lines perpendicular to a given line PQ is the pencil of circles perpendicular to the two intersecting circles PQ and Σ (see Section 3, p. 45f.). Since through each point in the plane there passes just one circle of this pencil, it follows that *from each point A of hyperbolic geometry one can drop a unique* (non-Euclidean) *perpendicular to a hyperbolic line PQ.*

Now we deal with the definition of distance between two points in hyperbolic geometry. We can proceed in much the same way as we did in the Supplement in NML 24 (see pp. 104–107). Specifically, in view of property D of inversion (p. 63; see also the remark on p. 72), circular transformations preserve the cross ratio of four points. We use this fact as follows. Suppose that a non-Euclidean motion takes points A and B to points A' and B', and that the non-Euclidean lines AB and $A'B'$ intersect the circle Σ at points P, Q and P', Q' respectively (Figure 129). Then

$$\frac{AP}{BP} : \frac{AQ}{BQ} = \frac{A'P'}{B'P'} : \frac{A'Q'}{B'Q'}.$$

On the other hand, if A, B, and C are three successive points on a non-Euclidean line that intersects Σ at P and Q, then, clearly,

$$\left(\frac{AP}{BP} : \frac{AQ}{BQ}\right) \times \left(\frac{BP}{CP} : \frac{BQ}{CQ}\right) = \left(\frac{AP}{CP} : \frac{AQ}{CQ}\right).$$

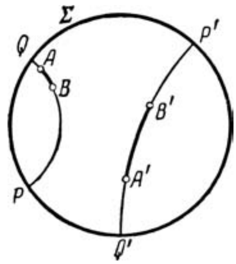


FIGURE 129

Hence, if we put

$$\log\left(\frac{AP}{BP} : \frac{AQ}{BQ}\right) = d_{AB}, \qquad (*)$$

then we obtain the equality

$$d_{AB} + d_{BC} = d_{AC},$$

a relation which we expect to hold for a measure of length of segments. *It therefore makes sense to call the number d_{AB} defined by* $(*)$ *the non-Euclidean length of the segment AB* (or the non-Euclidean distance between the points A and B).

If the point B on the non-Euclidean line PQ tends to P (Figure 129), then the cross ratio $\frac{AP}{BP} : \frac{AQ}{BQ}$ tends to infinity. This implies that in hyperbolic geometry *the length of a ray AP (and of a whole line PQ)* is infinite, although the line PAQ is represented by a finite arc of a circle. We can prove this in a purely geometric way. We consider the totality of non-Euclidean lines perpendicular to the given line PQ. From a Euclidean viewpoint, this is a pencil of circles perpendicular to the intersecting circles Σ and PAQ. Let S_1 be the circle of the pencil passing through B and let O_1 be its center (Figure 130). Since Σ is perpendicular to S_1, the reflection in S_1 takes Σ to itself, and therefore leaves $\mathbb{K}$ as a whole fixed. This means that this reflection is a non-Euclidean motion. Since the circle PAQ is also perpendicular to S_1, our inversion takes it to itself as well. Hence our inversion takes A to the second point B_1 of intersection of the line O_1A and the circle PAQ. Since the segments AB and BB_1 are interchanged by a non-Euclidean motion, their non-Euclidean lengths are the same: $d_{AB} = d_{BB_1}$. In much the same way we obtain the equalities

$$d_{AB} = d_{BB_1} = d_{B_1B_2} = d_{B_2B_3} = \cdots = d_{B_{n-1}B_n},$$

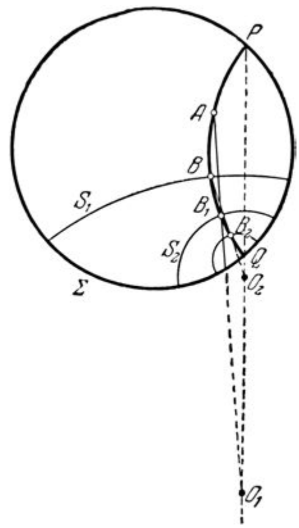


FIGURE 130

where B_2 is the point of intersection of PAQ with the line O_2B joining B to the center O_2 of the circle of the pencil passing through B_1; B_3 is the point of intersection of PAQ with the line O_3B_1 joining B_1 to the center O_3 of the circle of the pencil passing through B_2; and so on. This implies that if A is any point on a non-Euclidean line PAQ, then we can lay off any segment on it as many times as we wish without ever reaching its end Q.

What we have said so far shows how close hyperbolic geometry is to Euclidean geometry. In both geometries two points determine a unique line; a point and a ray issuing from it can be taken by a motion to any other point and any other ray issuing from it; on a given line we can lay off at any point in either direction a segment of arbitrary length (this is due to the infinite length of a ray); at any given point, beginning at any line through that point, we can lay off any given angle in either (clockwise or counterclockwise) direction, and so on. That is why all theorems of Euclidean geometry based on these very simple propositions carry over to hyperbolic geometry. This is true of the congruence tests for triangles, of theorems in which one compares the lengths of oblique and perpendicular line segments, of the fact that the angle bisectors in a triangle are concurrent, and so on (cf. pp. 115–117 of the Supplement in NML 24). The key difference between the two geometries is that *the parallel axiom does not hold in hyperbolic geometry.* In fact, it is easy to see that among the non-Euclidean lines passing through

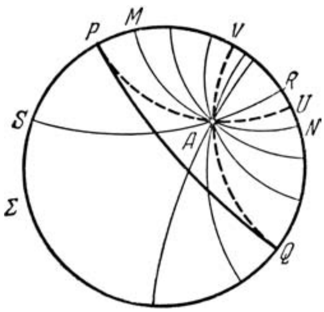


FIGURE 131

a point A not on a line PQ there are infinitely many that intersect it (such a line is the line RS in Figure 131) and infinitely many that do not intersect it (are ultraparallel to PQ; one such line is the line MN in Figure 131). The two classes of lines are separated by the two lines UP and VQ said to be parallel to PQ (cf. pp. 118 and 119 in NML 24).

77. Prove the following theorem of hyperbolic geometry: The angle bisectors in a triangle are concurrent. State the Euclidean version of this theorem.

78. Prove the following theorem of hyperbolic geometry: If the base angles in a triangle are equal then the triangle is isosceles. State the Euclidean version of this theorem.

79. (a) Let PQ and RS be two hyperbolic lines that intersect in a point B. Show that the distances from the points of RS to PQ (that is, the lengths of the perpendiculars dropped from the points of RS to PQ) increase beyond all bounds on both sides of B. Also, the feet of the perpendiculars dropped from the points of RS to PQ cover just a finite segment P_1Q_1 and the perpendiculars erected at P_1 and Q_1 are parallel to RS (see Figure 132a and the schematic Figure 115a on p. 120 in NML 24).

(b) Let PQ and RS be two parallel hyperbolic lines. Show that the distances from the points of UP to PQ decrease beyond all bounds in the direction of the ray AP (A is any point on UP) and increase beyond all bounds in the direction of the ray AU. The projection of UP to PQ is the ray Q_1P, and the perpendicular to PQ erected at Q_1 is parallel to UP (see Figure 132b and the schematic Figure 115b on p. 120 in NML 24).

(c) Show that the two ultraparallel hyperbolic lines PQ and MN have a unique common perpendicular KL, and that, conversely, two hyperbolic lines with a common perpendicular are ultraparallel. The distances from

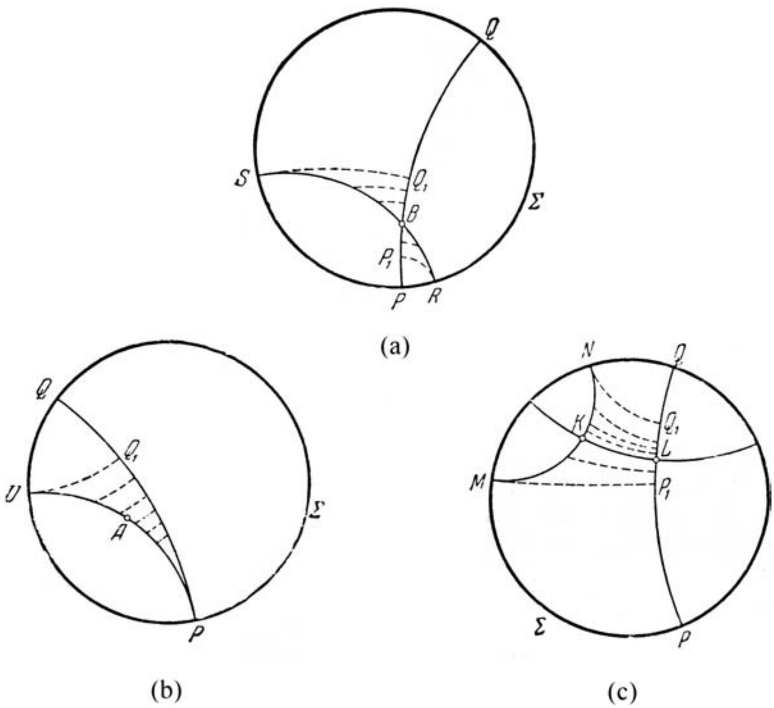


FIGURE 132

MN to PQ increase beyond all bounds on both sides of the foot K of their common perpendicular. The projection of MN to PQ is the finite segment P_1Q_1 of PQ, and the perpendiculars to PQ erected at P_1 and Q_1 are parallel to MN (see Figure 132c and the schematic Figure 115c on p. 120 in NML 24).

80. Show that in an acute hyperbolic triangle the altitudes are concurrent. Does this theorem hold for an obtuse-angled triangle?

81. Show that the sum of the angles in a hyperbolic triangle is less than 180°.

82. Show that two hyperbolic triangles with pairwise equal angles are congruent (that is, can be taken to one another by a hyperbolic motion).

Consider a pencil Π_1 of hyperbolic lines concurrent at A; in the Euclidean sense, this is the pencil of circles passing through A and $\overline{A}$ (compare Figures 133a and 6). Let $\overline{\Pi}_1$ be the pencil of circles perpendicular to the pencil Π_1 (see Section 3, p. 45); this is a pencil of nonintersecting

"ordinary" circles that includes Σ. A reflection in a circle S of the pencil Π_1 takes $\mathbb{K}$ to itself and is therefore a hyperbolic motion—a hyperbolic reflection in the line S (see pp. 125–126 in NML 24). Such a reflection takes a circle $\overline{S}$ of the pencil $\overline{\Pi}_1$ to itself; it follows that the product of reflections in two circles S_1 and S_2 of Π_1 is a hyperbolic rotation about A (see pp. 126–127 in NML 24)—it takes $\overline{S}$ to itself. The fact that any rotation about A takes all circles in the pencil $\overline{\Pi}_1$ to themselves implies that they are all hyperbolic circles centered at A, that is, loci of points equidistant (in the sense of hyperbolic geometry) from A. It follows that hyperbolic circles are ordinary (Euclidean) circles that do not intersect Σ; conversely, a circle $\overline{S}$ in the interior of Σ is a hyperbolic circle.[3]

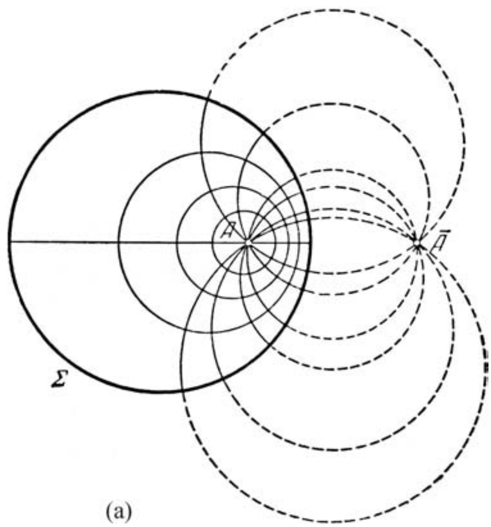
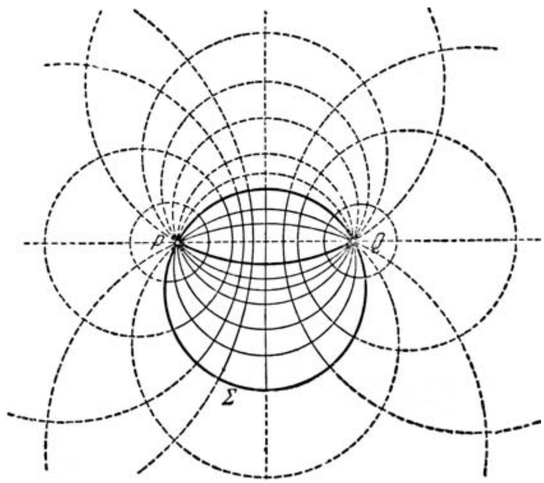


FIGURE 133

Now let Π_2 be a pencil of hyperbolic ultraparallels; this is a pencil of "ordinary" circles perpendicular to two intersecting (in fact, mutually perpendicular) circles PQ and Σ (Figure 133b). Let $\overline{\Pi}_2$ be the pencil of circles perpendicular to Π_2; this is a pencil of intersecting circles which includes PQ and Σ. A reflection in a circle S of Π_2—a hyperbolic reflection in the line S—takes all circles of $\overline{\Pi}_2$ to themselves. It follows that the product of two reflections in circles S_1 and S_2 of Π_2—a hyperbolic translation along the line PQ (see pp. 127–128 in NML 24)—takes all circles of $\overline{\Pi}_2$ to themselves. Hence every circle $\overline{S}$ of $\overline{\Pi}_2$ other than Σ and PQ is

a hyperbolic equidistant curve—briefly, an equidistant—with axis PQ, that is, the locus of points equidistant (in the sense of hyperbolic geometry) from PQ (see p. 128 in NML 24). Thus hyperbolic equidistants are circles intersecting Σ; conversely, a circle $\overline{S}$ intersecting Σ is an equidistant whose axis is the hyperbolic line passing through the points of intersection of $\overline{S}$ and Σ.



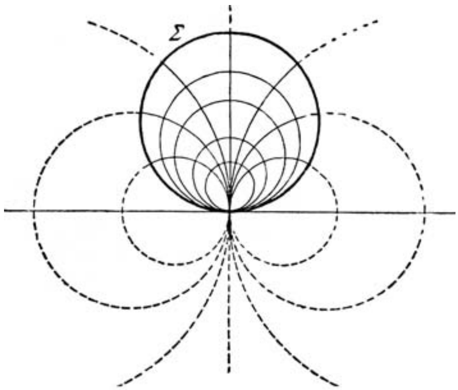
(b)

FIGURE 133

Finally, let Π_3 be a pencil of hyperbolic parallel lines, that is, a pencil of tangent circles perpendicular to Σ, and $\overline{\Pi}_3$ the pencil of circles tangent to Σ and perpendicular to Π_3 (Figure 133c). A reflection in a circle S of Π_3 —a hyperbolic reflection in the line S—takes all circles of $\overline{\Pi}_3$ to themselves. Hence these circles are limit lines of hyperbolic geometry or horocycles (see p. 129 in NML 24). Thus horocycles are circles tangent to Σ.

Circles, equidistants, and horocycles, as well as hyperbolic lines, are sometimes called cycles of hyperbolic geometry. The angle between two cycles S_1 and S_2 is defined as the angle between the two Euclidean circles S_1 and S_2. Cycles S_1 and S_2 are said to be tangent if the Euclidean circles S_1 and S_2 are tangent in the usual sense of the term.

83. Show that one can circumscribe a unique cycle (circle, equidistant, or horocycle) about any hyperbolic triangle.



(c)

FIGURE 133

84. Let A, B, C, and D be four nonconcyclic hyperbolic points, and let S_1, S_2, S_3, and S_4 be the cycles circumscribed about the triangles ABC, ABD, ACD, and BCD. Show that

(a) if two of the cycles intersect in a point P different from A, B, C, and D, then all four cycles pass through P;

(b) the angles between any two of the cycles S_1, S_2, S_3, and S_4 are equal to the angles between any other two of them.

85. Let S be a hyperbolic cycle and let A and B be two points of S. Draw all possible pairs of cycles S_1 and S_2 tangent to S at A and B and tangent to one another. Show that the locus of the points of tangency of these pairs of cycles is a cycle $\overline{S}$.

86. Let S_1, S_2, S_3, and S_4 be four hyperbolic cycles such that S_1 is tangent to S_2, S_2 to S_3, S_3 to S_4, and S_4 to S_1. Show that the four points of tangency are concyclic.

87. The totality of hyperbolic cycles perpendicular to two given cycles S_1 and S_2 is called a pencil of cycles. List all possible types of hyperbolic pencils of cycles. Show that if Π is a pencil of cycles, then there are infinitely many cycles perpendicular to all cycles of Π; these cycles form a new pencil $\overline{\Pi}$ said to be perpendicular to Π.

88. Let S_1 and S_2 be two hyperbolic cycles. Show that if there are circles perpendicular to S_1 and S_2, then their centers lie on a line l (see the schematic Figure 134; here S_1 is a circle and S_2 is an equidistant). The line l is called the radical axis of S_1 and S_2. If S_1 and S_2 intersect one another, then their radical axis is their common chord.

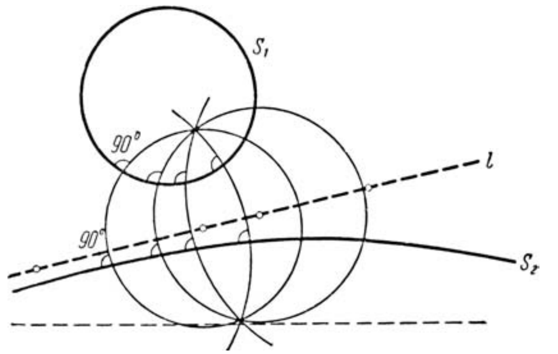


FIGURE 134

89. (a) Show that if two hyperbolic cycles perpendicular to a third cycle S and passing through a point A intersect in yet another point A', then all cycles perpendicular to S and passing through A pass through A'. The point A' is said to be symmetric to A with respect to S.

(b) The transformation that takes a hyperbolic point A to a point A' symmetric to A with respect to a cycle S is called a reflection in S. Show that a reflection in S takes cycles to cycles and preserves angles between cycles.

In the Supplement in NML 24 we noted that in addition to hyperbolic geometry there are other non-Euclidean geometries. One such is the so-called elliptic geometry.

We return for a moment to hyperbolic geometry. In our account of this geometry a key role was played by the circles perpendicular to a fixed circle Σ (in Section 3 we called this set a hyperbolic bundle of circles; see the text in fine print on p. 56); we took these circles as the lines of hyperbolic geometry. If such a line passes through a point A, then it it also passes through the point $\overline{A}'$, symmetric to A with respect to Σ; hence we can view A and $\overline{A}'$ as one and the same hyperbolic point. This is equivalent to eliminating the points of the plane in the exterior of Σ and constructing hyperbolic geometry in the interior of Σ (of course, one could have done the opposite, that is, one could have considered only the exterior of Σ—an approach that would have required replacing all figures in this Supplement by their reflections in Σ). We called a reflection in a circle S perpendicular to Σ a non-Euclidean reflection in the line S; here the essential thing is that such a reflection takes "hyperbolic lines" to "hyperbolic lines." Further, *hyperbolic motions can be defined as all possible products of reflections* (cf. pp. 125–126 of the Supplement in NML 24), and *hyperbolic geometry can be viewed as the study of the properties of figures preserved by hyperbolic motions defined in this way.*

Now we consider the totality of circles intersecting a fixed circle Σ in diametrically opposite points (in Section 3 we called this set an elliptic bundle). The totality of these circles which pass through a definite point A is a pencil of intersecting circles; the second point $\overline{A}$ of intersection of the circles of this pencil is obtained from A by the inversion with center O and power $-R^2$, where O is the center of Σ and R is its radius (compare Figure 135 with Figure 9b on p. 7). It follows that through every point in the plane, in any direction at that point, there passes just one such circle, and that any two points, not obtainable from one another by an inversion with center O and power $-R^2$, can be joined by just one such circle. We will call the circles intersecting Σ in diametrically opposite points, as well as Σ itself, lines of elliptic geometry and view points A and $\overline{A}$, obtained from one another by the inversion with center O and power $-R^2$, as the same point of elliptic geometry. In other words, the points of elliptic geometry are all points in the interior of Σ and half the points of Σ. (Two diametrically opposite points of Σ are obtained from one another by the inversion with center O and power $-R^2$, and thus must be identified. This being so, only one of the two semicircles of Σ, and only one of the two endpoints of that semicircle, belong to elliptic geometry.)

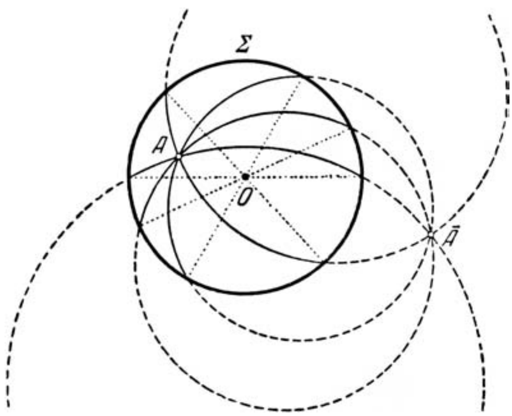


FIGURE 135

We will call a reflection in a Euclidean circle S a reflection in the line S of elliptic geometry. We will show that *such a reflection takes elliptic lines to elliptic lines*. In fact, let S_1 be an elliptic line, that is, a circle intersecting Σ in diametrically opposite points M and N, the points of intersection of S and S_1 (Figure 136; it is easy to see that S and S_1 must intersect one another). The fact that through M and N there pass two different circles which intersect Σ in two diametrically opposite points implies that these points are obtained by the inversion with center O and power $-R^2$. But then the circle S_1', symmetric to S_1 with respect to S, must also intersect Σ in two diametrically opposite points (for it passes through the points M and N), which is what we wished to prove. Now it is natural to define as the motions of elliptic

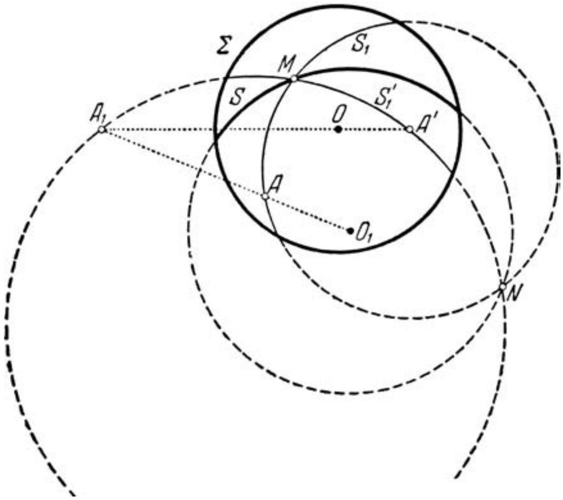


FIGURE 136

geometry all products of reflections in elliptic lines, and elliptic geometry as the study of properties of figures preserved by elliptic motions.[4]

The further development of elliptic geometry resembles the development of hyperbolic geometry in the early part of this Supplement. For example, consider the definitions, in elliptic geometry, of the distance between two points and the angle between two lines. It is clear that *the angle between two elliptic lines S_1 and S_2 can be defined as the Euclidean angle between the circles S_1 and S_2*. Two elliptic lines S_1 and S_2 are perpendicular (that is, the adjacent angles between them are equal) if the circles S_1 and S_2 are perpendicular in the usual sense. Now consider all perpendiculars erected at all points of an elliptic line $Q\overline{Q}$ (at every point one can erect a unique perpendicular, for through every point, in every direction, there passes a unique line). Let S_1 and S_2 be two such perpendiculars, and let P and $\overline{P}$ be the points of intersection of the circles S_1 and S_2 (Figure 137). Since through P and $\overline{P}$ there pass two circles S_1 and S_2 which intersect Σ in diametrically opposite points, these points are obtained from one another by the inversion with center O and power $-R^2$; since, in addition, the circles S_1 and S_2 are perpendicular to $Q\overline{Q}$, P and $\overline{P}$ are symmetric with respect to the circle $Q\overline{Q}$. It follows that every circle passing through P and $\overline{P}$ intersects Σ in diametrically opposite points and is perpendicular to $Q\overline{Q}$; hence these circles are the perpendiculars to $Q\overline{Q}$ erected at its different points. Since P and $\overline{P}$ must be regarded as a single elliptic point (only one of the two is in the interior of Σ), we conclude that *in elliptic geometry all perpendiculars to a line $Q\overline{Q}$ pass through a single point* (called the pole of the line $Q\overline{Q}$). Using this fact, *we can define the length d_{AB} of a segment AB of an elliptic line $Q\overline{Q}$ as the angle between the perpendiculars AP and BP to $Q\overline{Q}$ erected at the points A and B*; in fact, the magnitude d_{AB} defined in this way is preserved under the motions of elliptic geometry, and, if A, B, and C are three successive points of $Q\overline{Q}$, then

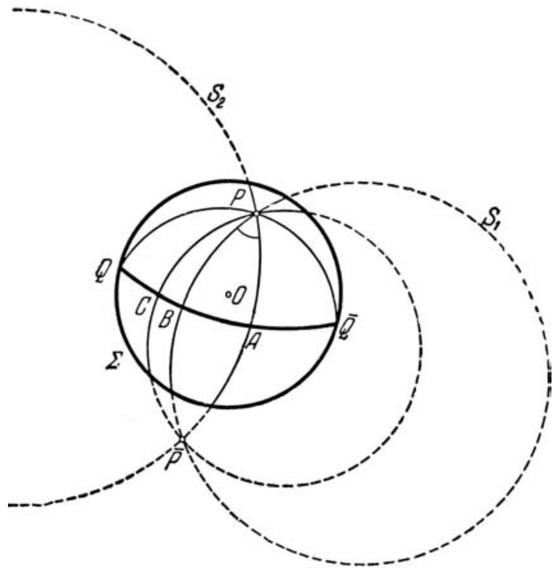


FIGURE 137

$d_{AB} + d_{BC} = d_{AC}$ (see Figure 137). Also, it turns out that an elliptic line $Q\overline{Q}$ has finite length (equal to the angle $QP\overline{Q}$ [5]); this fact underlies the profound difference between elliptic and Euclidean geometries (see the first footnote on p. 135 in NML 24).[6]

Earlier, we often made use of the fact that two circles which intersect Σ in diametrically opposite points intersect one another. In other words, any two elliptic lines intersect one another—*in elliptic geometry there are no parallel lines.* Hence we can call the product of two reflections in any two elliptic lines S_1 and S_2 a rotation. This motion takes the point A of intersection of S_1 and S_2 to itself; since the distance between two points is unchanged by a motion, every point B other than A moves on a non-Euclidean circle centered at A—a locus of points equidistant (in the sense of elliptic geometry) from A (Figure 138). Since reflections in the "non-Euclidean lines" S_1 and S_2 are reflections in the circles S_1 and S_2, and since such reflections take a circle perpendicular to S_1 and S_2 to itself, it follows that non-Euclidean circles centered at A coincide with the Euclidean circles perpendicular to S_1 and S_2; all such circles form a pencil Π of nonintersecting circles. Thus elliptic circles are Euclidean circles[7] (except for circles which intersect Σ in diametrically opposite points and Σ itself—these are "lines" rather than circles. Incidentally, in elliptic geometry a line may be viewed as a special case of a circle—it is the locus of points equidistant from the pole of the line).

Elliptic geometry has much in common with hyperbolic geometry. For example, in elliptic geometry

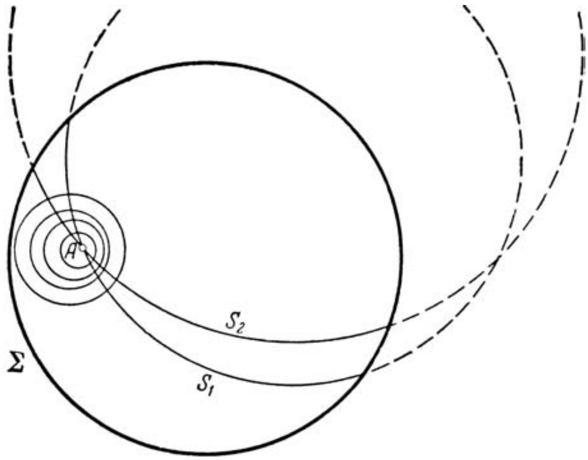


FIGURE 138

- the sum of two sides of a triangle is greater than the third side; the shortest distance between two points is a line segment;
- two triangles are congruent if: the sides of one are equal to the sides of the other; two sides of one and the angle between them are equal to two sides of the other and the angle between them; a side of one and the two angles adjacent to it are equal to a side of the other and the two angles adjacent to it; the angles of one are equal to the angles of the other (see problem 82);
- if the base angles in a triangle are equal then the triangle is isosceles (see problem 78); in an isosceles triangle the altitude, angle bisector, and median, drawn from the vertex on the equal sides, coincide;
- in any triangle, the perpendiculars erected at the midpoints of the sides intersect in a point—the circumcenter of the triangle; the angle bisectors intersect in a point—the incenter of the triangle (see problem 77); the altitudes are concurrent (see problem 80); the medians are concurrent (see problem 105 on p. 123 in NML 24);
- the sum of the angles in a triangle always exceeds 180° (see problem 81); the sum of the angles in an n-gon always exceeds $180° \cdot (n-2)$ (see problem 108b on p. 124 in NML 24).

The concept of angular excess is relevant in all three geometries. It is defined for a triangle as the difference $\angle A + \angle B + \angle C - 180°$. This difference is positive in elliptic geometry and negative in hyperbolic geometry (see problem 81); clearly, the sum of the angles in an n-gon exceeds $180° \cdot (n-2)$ in elliptic geometry and is less than $180° \cdot (n-2)$ in hyperbolic geometry (see problem 108b on p. 124 in NML 24).

There is a close connection between angular excess and area. Specifically, in elliptic geometry the area of a triangle is $k\ (\angle A + \angle B + \angle C - 180°)$, where

the value of k depends on the choice of unit of area (in hyperbolic geometry the relevant expression is $k\ (180° - \angle A - \angle B - \angle C)$). It is obvious how all this applies to Euclidean geometry.

The theorems in problems 83–89 carry over to elliptic geometry, except that here the term "cycles" stands for the circles and lines of this geometry.

The proofs of most of these theorems are close to the proofs of the corresponding theorems in hyperbolic geometry. For example, to show that the sum of the angles in an elliptic triangle exceeds 180° it suffices to take its vertex A by a non-Euclidean motion to the center O of the circle Σ. Then $\triangle ABC$ goes over to $\triangle OB'C'$ in Figure 139. It is clear from this figure that the sum of the angles of the curvilinear triangle $OB'C'$ is greater than the sum of the angles of the rectilinear triangle with the same vertices, that is, it is greater than 180° (see the solution of problem 81). Now the expression for the area of an elliptic polygon is deduced in a manner entirely analogous to the solution of problem 109, pp. 232–234, in NML 24. The reader should try to prove the theorems just stated.[8]

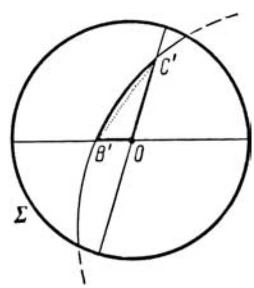


FIGURE 139

We note that if we define the circles that intersect in a point O as lines (circles of a parabolic pencil; see p. 56), and all products of reflections in these circles as motions, then we obtain Euclidean geometry (recall that the substratum of hyperbolic geometry is a hyperbolic pencil and the substratum of elliptic geometry is an elliptic pencil). In this connection see p. 161.

We conclude with comments in which we discuss the connection between our present account of hyperbolic geometry and the account in the Supplement on pp. 103–135 in NML 24. In these two accounts we seem to have given the same names to radically different objects (recall the two definitions of hyperbolic motions, of hyperbolic points and lines, of hyperbolic distances and angles, and of hyperbolic geometry itself). It is natural to ask for a justification of our modus operandi.

On p. 131 in NML 24 we noted that all theorems of hyperbolic geometry proved in that Supplement by considering points and chords of a disk $\mathbb{K}$ could also be proved differently, namely, in a manner analogous to that adopted in proving theorems of Euclidean geometry in school. The "usual" approach to proving theorems consists in reducing the required result to ever simpler ones until we reach the elementary "axioms" whose validity is accepted without proof. Clearly, different sets of initial axioms may yield different "geometries." Axioms specify basic properties of "points" and "lines" ("through two points there passes just one line," "on every line, in either direction from a point on that line, we can lay off a given segment any number of times," and so on) but do not tell us what points and lines are. True, we are used to associate with the words "points" and "lines" certain notions which help us find proofs by means of drawings (relying on this habit we suggested to the reader on p. 131 in NML 24 to illustrate theorems of hyperbolic geometry in which points and lines have unusual properties with schematic drawings, such as those in Figures 115 and 118), but, strictly speaking, reducing geometric propositions to axioms in no way calls for the use of drawings. Theorems can be proved without relying on drawings (but this makes proving them more difficult).

The nature of the basic geometric objects being, in principle, irrelevant in proving theorems, it is possible to produce different interpretations of one and the same geometric system. In fact, assume that we have been able to find a system of objects satisfying the very same relations as those required of points and lines. Then we can agree to call those objects "points" and "lines" and use them to construct a "geometry." Such a concrete interpretation of a geometry, characterized by a specific choice of axioms, is called a model, or representation, of the geometry in question. The same theorems hold in different models of the same geometry (and are deduced in the same way from the same axioms), but their concrete nature is radically different in different models.[9]

By now it is clear that in the Supplement in NML 24 and in the present Supplement we constructed two different models of the same—hyperbolic—geometry (which differs from Euclidean geometry in that it fails to satisfy the parallel axiom; see p. 119 in NML 24 and pp. 148–149 in the present account). The model constructed in the Supplement in NML 24 (points are the points of the disk $\mathbb{K}$; lines are line segments in the interior of $\mathbb{K}$; and motions are affine transformations which take $\mathbb{K}$ to itself) is called the Klein model (or the Klein-Beltrami model). The model in the present Supplement (points are the points of the disk $\mathbb{K}$; lines are circular arcs in

the interior of $\mathbb{K}$ that belong to circles perpendicular to the circle Σ which forms the boundary of $\mathbb{K}$; and motions are circular transformations which take $\mathbb{K}$ to itself) is known as the Poincaré model.

We note that one can construct different models of Euclidean geometry. Relevant instances follow.

Recall the principle of duality in Section 4 of NML 24. According to this principle, one can interchange in a geometric theorem the words "point" and "line" and the words "passes through" and "is incident on" and obtain in this way a true theorem. In other words, the principle of duality asserts the legitimacy of "renaming" geometric concepts. Specifically, one can call lines "points" and points "lines." The same section of NML 24 lists properties of polarities which provide additional rules for such renaming, such as the rule that the angle between two "lines" (that is, points) A and B is the angle AOB, where O is a certain fixed point in the plane (see property C of a polarity on p. 82 of NML 24). This interpretation of the words "point," "line," "angle," and so on, yields true theorems of the geometry, which means that the renaming procedure yields a new model of Euclidean geometry. All theorems of Euclidean geometry hold in this model but they express completely different geometric facts. This demonstrates the usefulness of the new model for proving geometric theorems (see problems 60, 62–64, and 69–74 on pp. 81 and 85 of NML 24).[10]

In Section 4 of the present book (pp. 71–72) we obtained another model of Euclidean geometry. Here we made use of the fact that inversion changes theorems to completely new theorems (see problems 53–59 in Section 4). This is equivalent to the existence of a definite model of Euclidean geometry obtained by means of inversions. In this model "points" are the points in the plane with the exception of a definite point O (and a fictitious "point at infinity" which is the image of O under inversion); "lines" are lines and circles passing through O; "circles" are lines and circles not passing through O; the "angle" between "lines" (that is, circles) S_1 and S_2 is the usual angle; the "distance" between points A and B is given by $AB \cdot \frac{k}{OA \cdot OB}$ (see the formula $(*)$ on p. 61); "reflection in a line" S is the reflection in the circle S; "motions" are products of "reflections in lines", and so on.[11]

An interesting model of three-dimensional Euclidean geometry is obtained by mapping it to the plane using a cyclographic projection (see Section 5, p. 129). In this model the "points" of space are the directed circles in the plane. This model can also be used to deduce new theorems; for example, it is easy to see that the theorem in problem 69(b) in Section 5 (p. 119) expresses in this model the well-known theorem which asserts that the medians in a triangle (disposed in space) meet in a point which divides them in the ratio 2:1 beginning at a vertex. If we adopt as the

"distance" between the "points" of this model (that is, between directed circles) the tangent distance, then we obtain a model of "pseudogeometry" of three-dimensional space (see pp. 131–134); then, for example, the formula that yields the solution of problem 69(a) yields the "pseudolength" of a median in a triangle (the reader should try to deduce this formula in a way similar to the deduction of the formula for the length of a median in a triangle in Euclidean geometry).

We repeat: our models of Euclidean geometry were obtained by using polarities (see Section 3 in NML 24) and inversions. These transformations enable us to establish a direct connection between any particular theorem of Euclidean geometry and its version in a model (see the problems from Section 3 in NML 24 and from Section 4 listed on pp. 71–72). We can establish this kind of direct connection between the Klein and Poincaré models of hyperbolic geometry; in other words, we can describe a transformation which changes either one of these models into the other. The details follow.

Let $\mathbb{K}$ be a disk of radius R in which we construct the Klein model of hyperbolic geometry and let σ be a sphere of radius R tangent to the plane $\mathbb{K}$ at the center of $\mathbb{K}$ (Figure 142). The orthogonal projection of the lower hemisphere of σ to the plane sets up a correspondence between the points of the hemisphere and the points of $\mathbb{K}$; under this correspondence, to the chords of $\mathbb{K}$, that is, to the "lines" of the Klein model, there correspond circular arcs of σ perpendicular to the circle Σ_1 that bounds the hemisphere (the "equator" of σ). Now, conversely, we map the hemisphere to the plane π by stereographic projection (see Section 3, p. 55 in NML 24). Its image is the disk $\mathbb{K}'$ of radius $2R$, and the images of the circular arcs perpendicular to the equator Σ_1 are circular arcs perpendicular to the circle Σ' of the disk $\mathbb{K}'$, that is, the "lines" of the Poincaré model. One can show (see the text in fine print below) that the "distance," in the sense of the Klein model, between two points of $\mathbb{K}$ is equal to the "distance," in the sense of the Poincaré

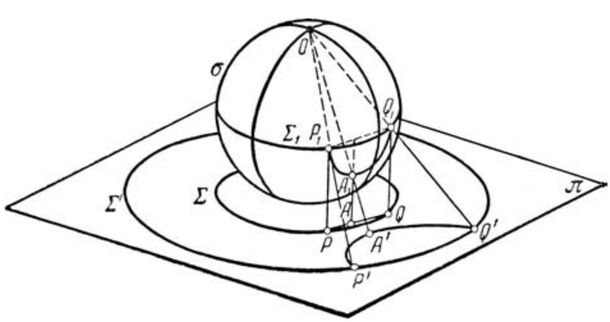


FIGURE 142

model, between their images in $\mathbb{K}'$, and that the "angle," in the sense of the Klein model, between two chords of $\mathbb{K}$ is equal to the angle between their images in $\mathbb{K}'$, namely, circular arcs perpendicular to $\mathbb{K}'$. In other words, *our transformation takes the Klein model to the Poincaré model.*

In the Supplement in NML 24 we defined hyperbolic geometry as the study of properties of points and chords of a disk $\mathbb{K}$ invariant under affine transformations that take $\mathbb{K}$ to itself—the "non-Euclidean motions" of the Klein model of non-Euclidean geometry investigated in that Supplement. In the present Supplement we defined hyperbolic geometry as the study of properties of points of a disk $\mathbb{K}$ and of circular arcs in the interior of $\mathbb{K}$ belonging to circles perpendicular to Σ, the boundary of $\mathbb{K}$, that are invariant under circular transformations which take $\mathbb{K}$ to itself—the "non-Euclidean motions" of the Poincaré model (see p. 104 in NML 24 and p. 143). Therefore, to prove that the transformation shown in Figure 142 takes the Klein model to the Poincaré model it suffices to prove that it takes the totality of affine transformations that fix $\mathbb{K}$ to the totality of circular transformations that fix $\mathbb{K}'$.[12] But it is clear that to every circular transformation that takes $\mathbb{K}'$ to itself there corresponds an affine transformation that takes $\mathbb{K}$ to itself; this follows from the fact that $\mathbb{K}$ goes over to $\mathbb{K}'$ and the lines intersecting $\mathbb{K}$ to circles perpendicular to Σ' (which a circular transformation fixing $\mathbb{K}'$ takes again to such circles). Further, among the "non-Euclidean motions" of the Poincaré model there are just two transformations that take to one another two given points A' and A_1' of $\mathbb{K}'$ with directions specified at these points;[13] therefore, among the affine transformations of $\mathbb{K}$ which correspond to these motions, there are just two transformations that take to one another two arbitrary points A and A_1 of $\mathbb{K}$ with directions specified at these points. This already implies that our transformation takes the totality of "non-Euclidean motions" of the Klein model to the totality of "non-Euclidean motions" of the Poincaré model.

We conclude by reproducing a brilliant transformation of the Klein model to the Poincaré model recently discovered by Ya. S. Dubnov; unlike the transformation just given, it does not involve space arguments. Let $\mathbb{K}$ be a disk with center O in which we construct the Poincaré model. With each point A of $\mathbb{K}$ we associate the point A' of the ray OA such that

$$d_{OA'} = \frac{1}{2} d_{OA}$$

(Figure 143); here d_{OA} and $d_{OA'}$ are non-Euclidean lengths of the segments OA and OA' (relying on the analogy with Section 1 in Chapter 1 of NML 21 we might call such a transformation a "non-Euclidean central similarity" with similarity coefficient $\frac{1}{2}$). We claim that this transformation takes a non-Euclidean line of the Klein model (that is, the chord PK of $\mathbb{K}$) to the non-Euclidean line PQ of the Poincaré model constructed in $\mathbb{K}$ (that is, to the arc PQ of the circle S perpendicular to the boundary circle Σ of $\mathbb{K}$), and the Klein model to the Poincaré model.

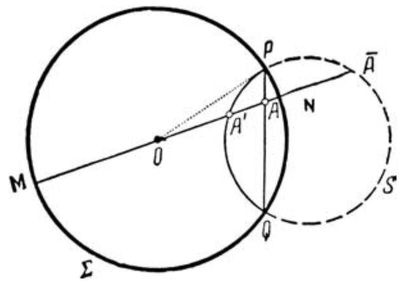


FIGURE 143

For proof denote by A the point of intersection of any diameter MN of $\mathbb{K}$ with the chord PQ, and by A' the point of intersection of that diameter with the arc PQ of the circle S; we must show that $d_{OA'} = \frac{1}{2}d_{OA}$. By the definition of the non-Euclidean length of a segment (see p. 107 in NML 24)

$$d_{OA} = \log\left(\frac{ON}{OM} : \frac{AN}{AM}\right) = \log\frac{AM}{AN} = \log\frac{R+OA}{R-OA} \qquad (*)$$

and

$$d_{OA'} = \log\left(\frac{ON}{OM} : \frac{A'N}{A'M}\right) = \log\frac{A'M}{A'N} = \log\frac{R+OA'}{R-OA'}; \qquad (**)$$

here R is the radius of $\mathbb{K}$. Further, let $\overline{A}$ be the second point of intersection of MN and S. Then $OA' \cdot O\overline{A} = R^2$, and therefore $O\overline{A} = \frac{R^2}{OA'}$. Now by a well-known property of chords of circles applied to Σ and S we have

$$MA \cdot AN = PA \cdot AQ = A'A \cdot A\overline{A},$$

or

$$(R+OA)(R-OA) = MA \cdot AN = A'A \cdot A\overline{A} = (OA - OA')(O\overline{A} - OA);$$

so that

$$R^2 - OA^2 = (OA - OA')\left(\frac{R^2}{OA'} - OA\right)$$

$$= OA \cdot \frac{R^2}{OA'} - R^2 - OA^2 + OA \cdot OA',$$

whence

$$OA = 2R^2 : \left(\frac{R^2}{OA'} + OA'\right) = \frac{2R^2 \cdot OA'}{R^2 + OA'^2}.$$

Finally, we obtain:

$$
\begin{aligned}
d_{OA} &= \log \frac{(R + OA)}{(R - OA)} \\
&= \log\left[\left(R + \frac{2R^2 \cdot OA'}{R^2 + OA'^2}\right) : \left(R - \frac{2R^2 \cdot OA'}{R^2 + OA'^2}\right)\right] \\
&= \log \frac{R^3 + R \cdot OA'^2 + 2R^2 \cdot OA'}{R^3 + R \cdot OA'^2 - 2R^2 \cdot OA'} \\
&= \log \frac{(R + OA')^2}{(R - OA')^2} = 2 \log \frac{R + OA'}{R - OA'} = 2d_{OA'},
\end{aligned}
$$

which was to be proved. We add that since formulas ($*$) and ($**$) could also be used to define the "non-Euclidean lengths" of the segments OA and OA' in the Poincaré model (see p. 65), it follows that in order to go, conversely, from the Poincaré model to the Klein model, we must apply to the Poincaré model a "non-Euclidean central similarity" with similarity coefficient 2.

From this there follows a simple construction that allows us to obtain the magnitude of a non-Euclidean angle in the Klein model. Let $P\overline{P}$ and $Q\overline{Q}$ be two intersecting lines in the Klein model, that is, two chords $P\overline{P}$ and $Q\overline{Q}$ of $\mathbb{K}$ that intersect in A (Figure 144). The transformation that takes the Klein model to the Poincaré model takes these lines to arcs $P\overline{P}$ and $Q\overline{Q}$ of circles S_1 and S_2 perpendicular to the boundary circle Σ of $\mathbb{K}$; construction of these circles is easy. Further, the non-Euclidean angle between the lines $P\overline{P}$ and $Q\overline{Q}$ of the Klein model (whose magnitude δ_{PAQ} is to be determined) goes over to the non-Euclidean angle between the lines S_1 and S_2 of the Poincaré model, that is, to the Euclidean angle between the circles S_1 and S_2. The latter is the angle between the tangents to S_1 and

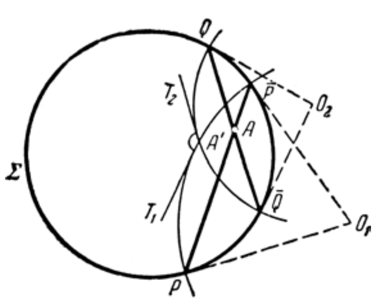


FIGURE 144

S_2 at their common pont A'. Hence

$$\delta_{PAQ} = \angle T_1 A' T_2.$$

It is clear that this construction is simpler than the one given in the Supplement in NML 24 (see pp. 112–114, and, in particular, Figure 110a).

Notes to Supplement

[1] We will see that all theorems of what we defined as hyperbolic geometry coincide with the theorems of the geometry introduced in the Supplement in NML 24. This justifies the use of the term "hyperbolic geometry" in both cases. A detailed discussion of the connection between the two systems is given on pp. 159–166.

[2] In the sequel we will, as a rule, use the term "circles" rather than "circles and lines." In cases in which "circles" includes lines ("circles of infinite radius") this will be clear from the context.

[3] However, one should keep in mind that the non-Euclidean center A of a circle $\overline{S}$ does not coincide with its Euclidean center $\overline{O}$. To find A we must consider the pencil of circles perpendicular to $\overline{S}$ and Σ; A is the point of intersection of the circles of this pencil contained in the interior of Σ.

[4] We note an essential difference between hyperbolic and elliptic motions. In hyperbolic geometry motions take the interior of Σ to itself, so that in this geometry we can ignore the points in the exterior of Σ. In contradistinction to this, in elliptic geometry motions do not take the interior of Σ to itself. This being so, we must always keep in mind the fact that points A and $\overline{A}$ obtained from one another by an inversion with center O and power $-R^2$ are to be viewed as a single point. For example, we must assume that a reflection in S takes the point A in Figure 136 to the elliptic point A' (reflection in S takes A to A_1 which we identify with A').

[5] It is easy to show that this angle is obtuse.

[6] An elliptic line is closed, like a circle (we recall that we agreed to think of Q and $\overline{Q}$ as a single point of a line). If, beginning at any point A, we lay off in an arbitrary direction a segment whose length is equal to that of a line, then we get back to A.

[7] One must keep in mind the fact that the non-Euclidean center A of S does not coincide with its Euclidean center O. To find A we must consider

the pencil of circles which intersect Σ in diametrically opposite points and are perpendicular to S (see p. 157, and, in particular, Figure 137); A is the point of intersection of the circles of this pencil located in the interior of Σ.

[8] In the Supplement in NML 24 we noted that the geometry of certain surfaces in three-dimensional space is hyperbolic. Similarly, the geometry of some other of these surfaces is elliptic. Such is spherical geometry. To establish the connection between elliptic geometry and spherical geometry we must map the sphere σ on the plane by stereographic projection (see Section 3, p. 55, in NML 24). Denote by Σ the image in the plane of the equator S of the sphere (Figure 140). The great circles on the sphere (the

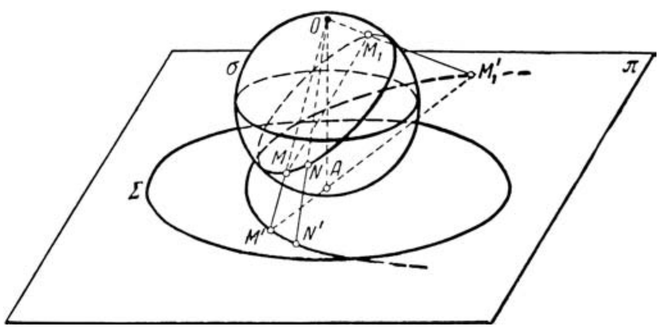


FIGURE 140

sections of the sphere by planes through its center) go over to plane circles which intersect Σ in diametrically opposite points. The angles between great circles on the sphere are the same as the angles between the plane circles that are their images (see Note 10 to Section 4, p. 78). If we adopt an appropriate unit of length, then the distance between two points M and N on the sphere (that is, the length of the shorter one of the two circular arcs determined by these points) is equal to the elliptic distance between their plane images M' and N'. To the rotations of the sphere about its center there correspond, under the stereographic projection, the elliptic motions of the plane. The condition that one should view points M' and M_1' in the plane, obtained from one another by the inversion with center O and power $-R^2$, as a single elliptic point translates into the need to identify the diametrically opposite points M and N on σ—this condition is due to the fact that two great circles on σ intersect in two diametrically opposite points. The connection between elliptic and spherical geometries can be used to advantage in proving theorems of elliptic geometry.

[9] An admittedly difficult book that discusses the issues touched on in the Supplement in NML 24 and in the present Supplement is V. I. Kostin's *Foundations of Geometry*, Moscow, Gos. ucheb.-pedagog. izd., 1948. (Russian)

[10] In Section 4 in NML 24 we noted that the principle of duality is of greatest value when applied to projective geometry; in other words, the "dual model" of projective geometry is of great convenience. As for the dual model of Euclidean geometry, that model has the defect that in it the term "point" applies to the lines in the plane and to a fictitious "line at infinity," and the term "line" applies to points in the plane and to the so-called "points at infinity," which may be thought of as directions.

[11] We emphasize that nothing is said in the axioms of geometry either about the nature of its fundamental objects ("points" and "lines") or about the sense of the basic relations between them (for example, about the sense of "equality" between segments, and "equality" between angles, or about the sense of the concept "a line passes through a point"); that is why, when we construct a model we must specify the sense of these relations between "points" and "lines." For example, in the Klein model of hyperbolic geometry "congruence" of figures means that there is an affine transformation that fixes $\mathbb{K}$ and takes one of the figures to the other, whereas in the Poincaré model the same term means that there is a circular transformation that fixes $\mathbb{K}$ and takes one of the figures to the other. If we agree to call "lines" not the chords of $\mathbb{K}$ but their poles with respect to Σ (see Section 4, pp. 66–67 in NML 24), then we obtain a new model in which "points" are the points of $\mathbb{K}$ and "lines" are the points exterior to $\mathbb{K}$, and in which the term "a line passes through a point" takes on an entirely unexpected meaning: a "line" (that is, a point) A passes through a point M if the chord joining the points of tangency with the circle Σ of the tangents from A to Σ passes through M (Figure 141).

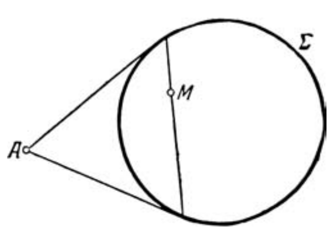


FIGURE 141

[12] This implies, for example, that the "non-Euclidean distance" between points A and B in the disk $\mathbb{K}$ in the Klein model coincides with the "non-Euclidean distance" in the Poincaré model between the points A' and B' in the disk $\mathbb{K}'$ that correspond to them (of course, since the units of length can be chosen independently in the two cases, the two "non-Euclidean distances" are proportional rather than equal). In fact, the "distance" d_{AB} between points A and B is characterized by the following conditions:

1° if the pair of points A, B goes over under a motion to the pair of points A_1, B_1 (the segments A, B and A_1, B_1 are "congruent"), then $d_{AB} = d_{A_1B_1}$;

2° if A, B, and C are three successive points on a line, then $d_{AB} + d_{BC} = d_{AC}$ (see the definition of "non-Euclidean distance" on pp. 104–107 in NML 24).

Hence the facts that the line AB goes over to the line $A'B'$ and "motions" go over to "motions" guarantee the equality of the distances d_{AB} and $d_{A'B'}$.

[13] These two transformations differ by the "non-Euclidean reflection" (see pp. 125–126 in NML 24) in the line passing through A'_1 in the direction given at A'_1.

Solutions

Section 1

1. Let the circle S be tangent to S_1 and S_2 at A and B; let O_1, O_2, and $\overline{O}$ be the centers of S_1, S_2, and S and let O be the point of intersection of AB and O_1O_2 (Figure 145). We apply the inversion with center O and power $k = OA \cdot OB$. Then A and B are interchanged; S goes over to itself (for any line through O intersects S at M and N such that $OM \cdot ON = OA \cdot OB = k$); and S_1, tangent to S at A, goes over to S_1', tangent to S at B. Also, the center of S_1' lies on the line OO_1 (see the proof of property B_4, p. 9), that is, it coincides with the point O_2 of intersection of $\overline{O}B$ and OO_1. It follows that S_1' coincides with S_2. Hence the inversion with center O takes S_1 to S_2, that is, O is the center of similarity of S_1 and S_2 (see the proof of property B_4); and this is what was to be proved.

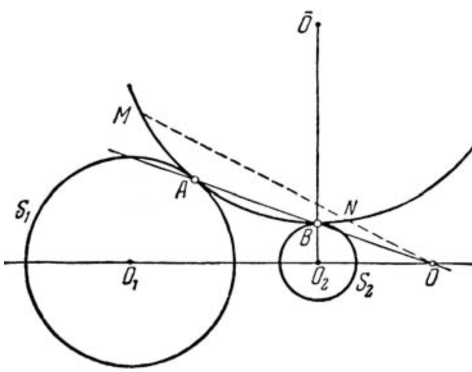


FIGURE 145

[If AB coincides with O_1O_2, our proposition is obvious. If $AB \parallel O_1O_2$, then, instead of the inversion with center O, one should apply the reflection that takes A to B, in which case one shows, as above, that this reflection takes S_1 to S_2. Hence these circles are congruent and have no outer center of similarity; this is an exceptional case.]

2. (a) *First solution.* We apply an inversion with center at a point O of the circle S. Then S goes over to a line l and the circles S_1 and S_2 go over to circles S_1' and S_2' tangent to one another and to the line l at fixed points A' and B' (Figure 146a). Let M' be the point of tangency of S_1' and S_2' and let O' be the point of intersection of l and the line through M' tangent to S_1' and S_2'. Then, clearly,

$$O'A' = O'M' \quad \text{and} \quad O'B' = O'M',$$

that is,

$$O'A' = O'B' = O'M'.$$

Thus M' is a point of the circle with center at O' and radius equal to half the length of $A'B'$. It follows that the locus of points of tangency of S_1 and S_2 is the circle through A and B perpendicular to the circle S.

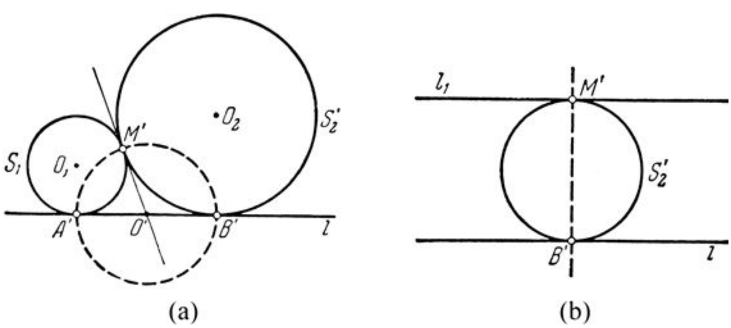


FIGURE 146

Second solution. One can simplify things by applying an inversion with center at A. Then the circle S goes over to a line l, and the circles S_1 and S_2, tangent to S at A and B, go over to a line l_1, parallel to l, and a circle S_2' tangent to l at the fixed point B' (Figure 146b). Clearly, the locus of points M' of tangency of l_1 and S_2' is a line perpendicular to l and l_1 and passing through B'. It follows that the locus of points M of tangency of S_1 and S_2 is a circle (passing through A and B and perpendicular to S).

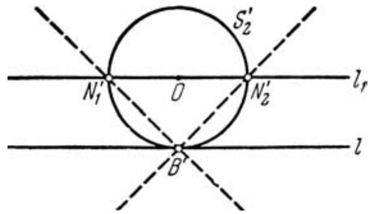


FIGURE 147

(b) Under an inversion with center at A the circle S goes over to a line l, the circle S_2 goes over to a circle S_2' tangent to l at a fixed point B', and the circle S_1 goes over to a line l_1 parallel to l and perpendicular to the circle S_2', that is, passing through its center (Figure 147). It is clear that the locus of points N_1' and N_2' of intersection of l_1 and S_2' consists of two (mutually perpendicular) circles passing through A and B' and forming with l a 45° angle.

3. We apply an inversion with center at A. Then the circles circumscribed about the triangles ABC, ABD, and ADC go over to lines and we obtain Figure 148. This figure shows that the angle between the circle S' circumscribed about the triangle $B'C'D'$ and the line $C'D'$ is equal to the angle between the lines $B'C'$ and $B'D'$ (each is equal to half the angle subtended by the arc $C'D'$ of S'). This implies the assertion of the problem.

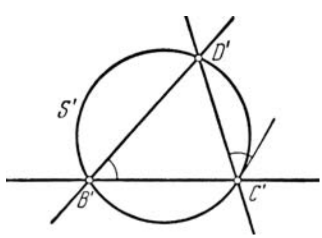


FIGURE 148

4. (a) Let r_1, r_2, and r be the radii of the circles S_1, S_2, and S; $r = r_1 + r_2$. Under the inversion with center M and (negative!) power $k = MA \cdot MB$ the circle S and the line MD go over to themselves, the circles S_1 and S_2 go over to tangents to S at the points B and A, and the circles Σ_1 and Σ_2 go over to circles Σ_1' and Σ_2' with radii r_1 and r_2; see Figure 149. The circles Σ_1 and Σ_1' are centrally similar with center M and coefficient of similarity $\frac{k}{k_1}$, where $k = MA \cdot MB = 4r_1r_2$ and k_1 is the square of

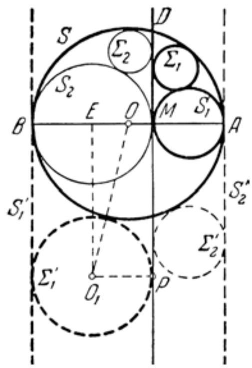


FIGURE 149

the tangent MP from M to Σ_1' (see the proof of property B_4, p. 9); from triangle OO_1E it follows that

$$MP^2 = O_1E^2 = OO_1^2 - OE^2 = (r + r_2)^2 - (r - r_2)^2 = 4rr_2.$$

This implies that the radius of Σ_1 is

$$r_2\frac{k}{k_1} = r_2\frac{4r_1r_2}{4rr_2} = \frac{r_1r_2}{r_1 + r_2}.$$

In just the same way we prove that the radius of Σ_2 is also $\frac{r_1r_2}{r_1+r_2}$.

(b) An inversion with center B and (positive!) power $BM \cdot BA$ takes the circle S_1 to itself, the circle S to the line MD, and the line MD to S. Thus the curvilinear triangle AMD in Figure 19 goes over to itself, and so does the circle Σ_1 inscribed in that triangle.

The point T of tangency of S_1 and Σ_1 stays fixed (for S_1 and Σ_1 go over to themselves); hence T is on the circle of inversion. The point T_1 of tangency of S_1 and the tangent to S_1 from B is also fixed (for S_1 and BT_1 go over to themselves); hence T_1 is also on the circle of inversion. But the semicircle S_1 intersects the circle of inversion in just one point; this implies that T_1 coincides with T and the tangent to S_1 and Σ_1 at T passes through B. We prove in the same way that the tangent to S_2 and Σ_2 at their point of tangency passes through A.

5. An inversion with center at A, the point of tangency of S_1 and S_2, transforms Figure 20 into Figure 150; clearly all we need show is that B', C', and D' in Figure 150 lie on the same line Σ'. Let MN be a common tangent to S_3' and S_4' at C', and let D_1 be the point of intersection of the lines $B'C'$ and S_1'. Then $\angle NB'C' = \angle NC'B'$ (each of them is equal to half the

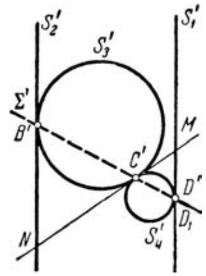


FIGURE 150

angle subtended by the arc $B'C'$ of the circle S_3'), $\angle NC'B' = \angle MC'D_1$ (vertical angles), and $\angle NB'C' = \angle MD_1C'$ (alternate interior angles with respect to the parallel lines S_1' and S_2'). Then $\angle MC'D_1 = \angle MD_1C'$, so $MD_1 = MC'$; and since $MD' = MC'$ (as tangents from M to S_4'), it follows that D_1 coincides with D', which was to be shown.

6. (a) We apply an inversion with center P, the point of intersection of the circles circumscribed about the circles $A_1A_2B_3$, $A_1A_3B_2$, and $A_2A_3B_1$. Then these three circles go over to lines and we end up with the following problem: *Show that the circles circumscribed about the triangles $B_1'B_2'A_1'$, $B_1'B_3'A_2'$, and $B_2'B_3'A_1'$, where B_1', B_2', and B_3' are points on the sides $A_2'A_3'$, $A_3'A_1'$, and $A_1'A_2'$ of the triangle $A_1'A_2'A_3'$, meet in a single point*, that is, problem 58(a) in NML 21.

(b) This problem can be formulated as follows. *Given six points A_1, C_1, D_2, C_2, A_2, B_1; show that if the circles circumscribed about the triangles $A_1C_1B_1(\Sigma)$, $C_1D_2C_2(S_4)$, and $D_2A_1A_2(S_1)$ meet in a point (D_1), then the circle circumscribed about the triangle $C_2A_2D_2(\Sigma')$ passes through the point (B_2) of intersection of circles circumscribed about the triangles $A_2A_1B_1(S_2)$ and $B_1C_2C_1(S_3)$*. But this is problem (a).

(c) An inversion with center O takes circles S_1, S_2, S_3, and S_4 to four lines; thus our problem becomes the following problem: *Show that four circles, circumscribed about four triangles formed by four pairwise intersecting lines* (no three of which meet in a point), *meet in a point*. This is problem 35 on p. 44 in NML 21.

7. (a) For $n = 4$ this proposition coincides with the one in problem 35 in NML 21. Now we assume that the proposition has been proved for all values of n less than a certain fixed value and prove it for that value. This approach to the solution of the problem, based on the method of mathematical induction, is suggested by its very formulation.

Assume that n ($\geqq 5$) is odd. We have n lines $l_1, l_2, l_3, \dots, l_n$; to each set of $n-1$ of these lines, obtained by removing line l_i $(i = 1, 2, \dots, n)$, there corresponds the central point A_i of these $n-1$ lines; to each set of $n-2$ of these lines, obtained by removing lines l_i and l_j $(i, j = 1, 2, \dots, n)$, there corresponds the central circle S_{ij} of these $n-2$ lines; to each set of $n-3$ of these lines, obtained by removing lines l_i, l_j, and l_k $(i, j, k = 1, 2, \dots, n)$, there corresponds the central point of these $n-3$ lines; to each set of $n-4$ of these lines, obtained by removing lines l_i, l_j, l_k, and l_m $(i, j, k, m = 1, 2, \dots, n)$, there corresponds the central circle S_{ijkm} of these $n-4$ lines (if $n = 5$ then, for example, instead of circle S_{1234} we have line l_5). We are to show that the n points $A_1, A_2, \dots, A_n$ are concyclic; for this it suffices to show that each four of these points, say, the points A_1, A_2, A_3, and A_4, are concyclic.

According to the definition of central points and central lines, A_1 is the point of intersection of the circles $S_{12}, S_{13}, S_{14}, \dots, S_{1n}$, and similar statements hold for $A_2, A_3, \dots, A_n$; the points $A_{123}, A_{124}, \dots, A_{12n}$ are on the circle S_{12}, and so on; A_{123} is the point of intersection of the circles $S_{1234}, S_{1235}, \dots, S_{123n}$, and so on. Thus we see that

the circles S_{12} and S_{23} intersect in points A_2 and A_{123};
the circles S_{23} and S_{34} intersect in points A_3 and A_{234};
the circles S_{34} and S_{41} intersect in points A_4 and A_{134};
the circles S_{41} and S_{12} intersect in points A_1 and A_{124}.

Since the four points A_{123}, A_{234}, A_{134}, and A_{124} are on the circle S_{1234}, it follows by the proposition in problem 6(b) that A_1, A_2, A_3, and A_4 are also concyclic, which is what we wished to prove.

Now we consider the case of an even n. We use notations analogous to those above except that now S_1 is the central circle of $n-1$ lines $l_2, l_3, \dots, l_n$, A_{12} is the central point of $n-2$ lines $l_3, l_4, \dots, l_n$, and so on. We are to prove that the n circles $S_1, S_2, \dots, S_n$, meet in a point; for this it suffices to prove that every three of them, say, S_1, S_2, and S_3 meet in a point.[1]

The definitions of central points and central circles imply that

the circle S_1 passes through the points A_{12}, A_{13}, and A_{14};
the circle S_3 passes through the points A_{13}, A_{23}, and A_{34};
the circle S_2 passes through the points A_{23}, A_{12}, and A_{24};
the circle S_{134} passes through the points A_{14}, A_{34}, and A_{13};
the circle S_{234} passes through the points A_{34}, A_{24}, and A_{23};
the circle S_{124} passes through the points A_{24}, A_{14}, and A_{12}.

But the last three of the above circles meet at the point A_{1234}; and therefore, in view of the proposition in problem 6(a), the first three circles also meet in a point, which is what we had to prove.[2]

(b) For the case $n = 3$ our assertion coincides with problem 58(a) on p. 75 in NML 24, and for $n = 4$ it is implied by the proposition in problem 6(b) (see the formulation for this case in Note 11, Section 1). Now we assume that the proposition in the problem has been proved for all n less than a certain particular one and we prove it for that value of n.

Assume that n ($\geqq 5$) is odd. We have n lines $l_1, l_2, l_3, \ldots, l_n$ and we choose a point on each of them (the points are concyclic); to each $n-1$ of these lines, obtained by removing line l_i, there corresponds the directing circle S_i; to each $n-2$ of these lines, obtained by removing lines l_i and l_j, there corresponds the directing point A_{ij}; to each $n-3$ of these lines, obtained by removing lines l_i, l_j and l_k $(i, j, k = 1, 2, \ldots, n)$, there corresponds the directing circle S_{ijk}; to each $n-4$ of these lines, obtained by removing lines l_i, l_j, l_k, and l_m $(i, j, k, m = 1, 2, \ldots, n)$, there corresponds the directing point A_{ijkm} (if $n = 5$ then, instead of, say, A_{1234}, there enters a point taken on the line l_5). We must show that the n circles $S_1, S_2, \ldots, S_n$ meet in a point; for this it suffices to show that any three of these circles, say, S_1, S_2, and S_3, meet in a point. The definitions of directing points and directing circles imply that

the circle S_1 passes through the points $A_{12}, A_{13},$ and A_{14};
the circle S_3 passes through the points $A_{13}, A_{23},$ and A_{34};
the circle S_2 passes through the points $A_{23}, A_{12},$ and A_{24};
the circle S_{134} passes through the points $A_{14}, A_{34},$ and A_{13};
the circle S_{234} passes through the points $A_{34}, A_{24},$ and A_{23};
the circle S_{124} passes through the points $A_{24}, A_{14},$ and A_{12}.

For the rest of the argument see the concluding part of the argument in problem 6(a).

The argument in the case of an even n is very similar to the argument in the first part of the solution of problem (a) above.

8. (a) Let S and s be the circumcircle and incircle of a triangle ABC, O and o their centers, R and r their radii, D, E, and F the points at which s touches the sides of the triangle, M, N, and P the points of intersection of the sides of the triangle DEF with oA, oB, and oC; obviously, these points are coincident with the midpoints of the sides of $\triangle DEF$ (Figure 151). We will prove that under a reflection in the circle S the points A, B, C go

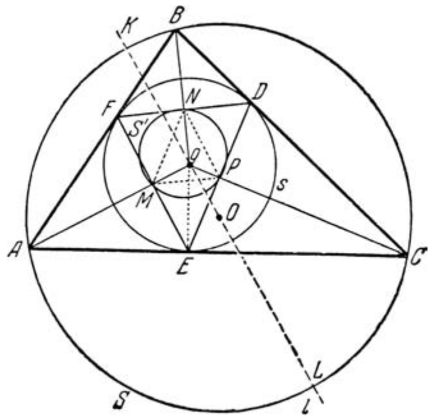


FIGURE 151

over to the points M, N, P. In fact, for example, the similarity of the right triangles oME and oEA implies that $\frac{oA}{oE} = \frac{oE}{oM}$, or $oA \cdot oM = oE^2 = r^2$. It follows that our inversion interchanges A and M.

Now we see that our inversion takes the circumcircle S of $\triangle ABC$ to the circumcircle S' of $\triangle MNP$. Since triangles MNP and DEF are similar with similarity coefficient $\frac{1}{2}$, it follows that the radius of S' is $\frac{r}{2}$. On the other hand, S' and S are centrally similar with center of similarity o and similarity coefficient $\frac{-r^2}{k}$, where $k = oK \cdot oL$ and K and L are the points of intersection with S of any line l passing through o (see the proof of property B_4 of inversion). Take as l the line oO; then $oK = R - d$, $oL = R + d$, and therefore $k = (R-d)(R+d) = R^2 - d^2$.

We see that S' and S, with radii $\frac{r}{2}$ and R, are centrally similar with similarity coefficient $\frac{r^2}{R^2-d^2}$. Hence

$$\frac{\frac{r}{2}}{R} = \frac{r^2}{R^2-d^2}, \quad \text{or} \quad \frac{1}{r} = \frac{(R-d)+(R+d)}{R^2-d^2} = \frac{1}{R+d} + \frac{1}{R-d},$$

which is what was to be proved.

The following is the simplest way to prove the converse proposition. Let R and r be the radii of circles S and s. Let the distance d between their centers be connected with R and r by the relation

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

This implies, first of all, that

$$2Rr = R^2 - d^2, \quad d^2 = R^2 - 2Rr < R^2 - 2Rr + r^2 = (R-r)^2, \quad d < R - r,$$

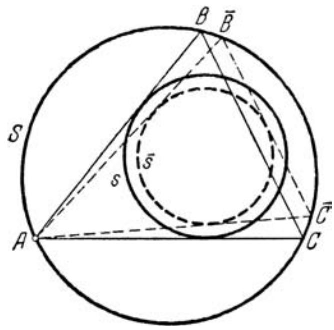


FIGURE 152

that is, s is completely contained in the interior of S. Let A be a point of S; we draw chords AB and AC of S tangent to s and join B to C (Figure 152). We assume that the line BC is not tangent to s but, say, cuts it. Without changing the center of s we shrink its radius until we obtain a circle $\bar{s}$ such that the chord $\overline{B}\,\overline{C}$ of S, where $A\overline{B}$ and $A\overline{C}$ are chords of S tangent to $\bar{s}$, is itself tangent to $\bar{s}$ (Figure 152). In view of what has been proved, the radius $\bar{r}$ of $\bar{s}$ and the magnitudes R and d are connected by the relation

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{\bar{r}}.$$

But this is impossible, for $\bar{r} < r$ and

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

In much the same way we prove that the chord BC cannot pass outside s (for this we would have to increase s).

(b) Proceeding in a manner analogous to the solution of problem (a) we show that a reflection in the excircle s_1 of $\triangle ABC$ takes its vertices to the midpoints M_1, N_1, and P_1 of the sides of $\triangle D_1E_1F_1$ with vertices at the points of tangency of s_1 to the sides of $\triangle ABC$ (Figure 153). It follows that the circumscribed circle S goes over to a circle S_1' of radius $\frac{r_1}{2}$ (cf. the solution of problem (a)). On the other hand, the circles S_1' and S are centrally similar with center of similarity o_1 and similarity coefficient $\frac{r_1^2}{k}$, where

$$k = o_1K_1 \cdot o_1L_1 = (d_1 - R)(d_1 + R) = d_1^2 - R^2$$

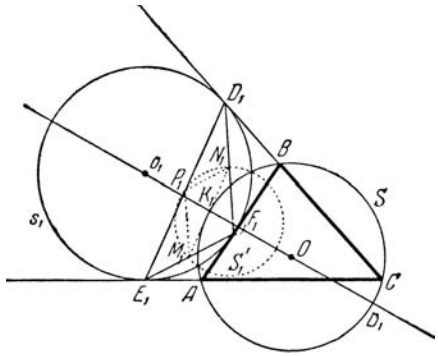


FIGURE 153

(see Figure 153). Hence

$$\frac{\frac{r_1}{2}}{R} = \frac{r_1^2}{d_1^2 - R^2},$$

and therefore

$$\frac{1}{d_1 - R} - \frac{1}{d_1 + R} = \frac{1}{r_1},$$

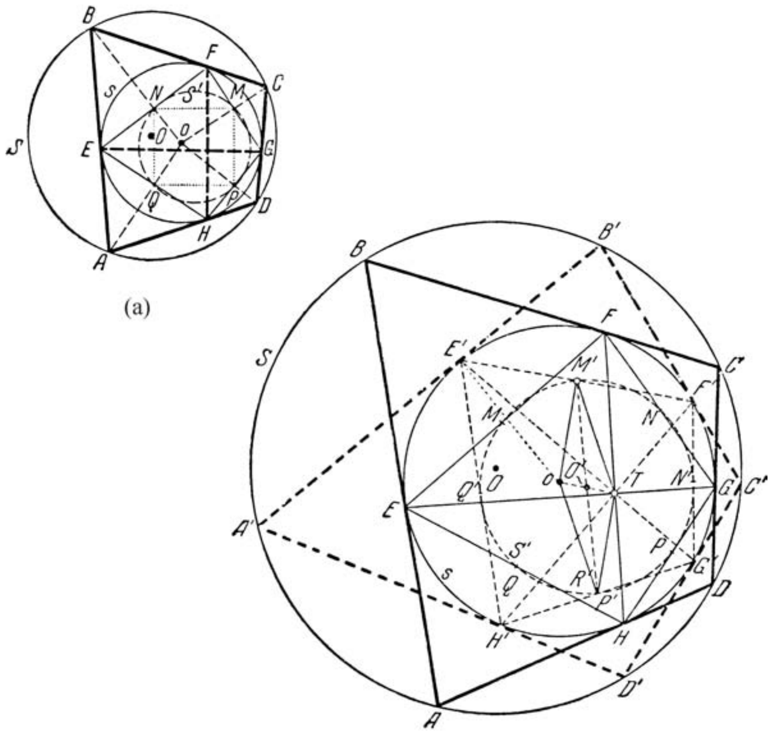
which is what was to be proved.

Remark. One can show that, conversely, if the radii R and r_1 of two circles and the distance d_1 between their centers are connected by the relation

$$\frac{1}{d_1 - R} - \frac{1}{d_1 + R} = \frac{1}{r_1},$$

and the first circle is not completely contained inside the second circle (this is not implied by the problem and must be separately stipulated), then these circles may be viewed as the circumcircle and excircle of a certain triangle (in fact, of infinitely many triangles; as a vertex of such a triangle we can take any point of the first circle in the exterior of the second circle).

9. (a) Let the quadrilateral $ABCD$ be inscribed in the circle S (with center O and radius R) and circumscribed about the circle s (with center o and radius r); let E, F, G, H be the points at which s touches the sides of the quadrilateral, and let M, N, P, Q be the points of intersection of oC, oB, oA, and oD with FG, EF, HE, and GH (the midpoints of the sides of the quadrilateral; Figure 154a). Reflection in s takes the vertices of $ABCD$ to the vertices of $MNPQ$ and the circle S to the circle S' circumscribed about $MNPQ$ (cf. the solution of problem 8(a)). But MN and PQ are midlines of the triangles EFG and GEH, so that $MN \parallel PQ \parallel EG$; we show similarly that $MP \parallel NQ \parallel FH$. Hence $MNPQ$ is a parallelogram



(a)

(b)

FIGURE 154

whose sides are parallel to FH and EG. Since the circle S' is circumscribed about this parallelogram, it follows that the latter is a rectangle. But then $FH \perp EG$, which is what we had to prove.

(b) We prove first that if there is a single quadrilateral inscribed in a given circle S and circumscribed about another circle s, then there are infinitely many such quadrilaterals. We saw that $FH \perp EG$ (Figure 154b; we use the same lettering as in problem (a)). Through the point T of intersection of FH and EG we pass another pair of mutually perpendicular chords $F'H'$ and $E'G'$ of s. We claim that the midpoints M', N', P' and Q' of the sides of the quadrilateral $E'F'G'H'$ lie on the same circle S' that appeared in the solution of problem (a).

All we need to show is that *the locus of midpoints of chords $E'F'$ of s such that $TE' \perp TF'$ consists of points located on a single circle $\overline{S}'$*; if so, then the fact that M, N, P and Q belong to this locus implies that $\overline{S}'$

coincides with S', which, in turn, implies that M', N', P' and Q' belong to this locus, and thus the required proposition. We join M' to T and to the center o of s and consider the parallelogram $M'TR'o$. The well-known property of a parallelogram implies that

$$M'R'^2 + To^2 = 2TM'^2 + 2oM'^2,$$

or

$$O'M'^2 + O'T^2 = \frac{1}{2}(TM'^2 + oM'^2),$$

where O' is the midpoint of To. Since TM' is a median of the right triangle $TE'F'$, it follows that $TM' = M'E' = M'F'$; since M' is the midpoint of the chord $E'F'$ of s, it follows that $OM' \perp E'F'$. Hence

$$oM'^2 + TM'^2 = oM'^2 + M'E'^2 = oE'^2 = r^2$$

(r is the radius of s), that is,

$$O'M'^2 = \frac{r^2}{2} - O'T^2$$

does not depend on the choice of the chord $E'F'$. This completes the proof of our assertion: the required locus is the circle with center O' and radius

$$\sqrt{\frac{r^2}{2} - O'T^2}.$$

Now let $A'B'C'D'$ be the quadrilateral formed by the tangents to s at E', F', G', and H'. As in the solution of problem 8(a), we show that A', B', C', D' are symmetric to M', N', P', Q' with respect to s. And since, by what has been proved, the latter points are on S', the vertices of $A'B'C'D'$ are on S (symmetric to S' with respect to s); it follows that the quadrilateral $A'B'C'D'$ is both circumscribed about s and inscribed in S.

Now we can deal with the main assertion of the problem. Let the line $E'G'$ coincide with the line oO of centers of the circles s and S. Considerations of symmetry show readily that in this case the quadrilateral $A'B'C'D'$ is a trapezoid (Figure 155). Since this trapezoid is inscribed in a circle, the sum of the bases is equal to the sum of the lateral sides, that is, the median of the trapezoid is equal to a lateral side. Therefore, if o is the midpoint of $E'G'$ (o is the center of s; $oE' = oG' = r$) and K is the midpoint of the lateral side $A'D'$, then the median oK of the triangle $oA'D'$ is equal to half the side $A'D'$; hence $oA'D'$ is a right triangle. This implies that $\angle A'oE' + \angle D'oG' = 90°$, that is, the right triangles $oA'D'$ and $oD'G'$ are similar.

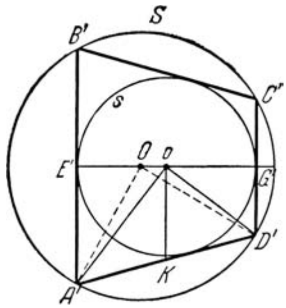


FIGURE 155

But if O is the center of S, $oO = d$, then

$$OE' = r - d, \quad OG' = r + d,$$
$$A'E' = \sqrt{OA'^2 - OE'^2} = \sqrt{R^2 - (r-d)^2},$$
$$D'G' = \sqrt{OD'^2 - OG'^2} = \sqrt{R^2 - (r+d)^2},$$

and the similarity of the triangles $oA'E'$ and $oD'G'$ implies that

$$\frac{r}{\sqrt{R^2 - (r-d)^2}} = \frac{\sqrt{R^2 - (r+d)^2}}{r},$$

whence

$$r^4 = (R^2 - r^2 - d^2)^2 - 4r^2d^2,$$
$$r^4 = (R^2 - d^2)^2 - 2r^2R^2 + r^4 - 2r^2d^2,$$
$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2).$$

If we rewrite the right side of the last equality as $r^2[(R+d)^2 + (R-d)^2]$ and divide both sides of the new version of that equality by $r^2(R^2 - d^2)^2 = r^2(R+d)^2(R-d)^2$, then we obtain the required relation

$$\frac{1}{r^2} = \frac{1}{(R-d)^2} + \frac{1}{(R+d)^2}.$$

[We have not shown that any point of the circle S can be chosen as a vertex of a quadrilateral inscribed in S and circumscribed about s; but this follows easily from the solution of the problem, or can be proved by an argument analogous to the argument in the solution of problem 8(a).]

10. The equality

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}$$

(see problem 8(a)) implies that

$$d^2 = R^2 - 2Rr = R(R - 2r),$$

so that

$$R - 2r \geq 0 \quad \text{and} \quad r \leq \frac{R}{2},$$

which is what was to be shown.

If $r = \frac{R}{2}$, then $d = 0$, that is, the excircle and incircle of the triangle are concentric (the point of intersection of the perpendiculars erected at the midpoints of the sides coincides with the point of intersection of the angle bisectors of the triangle). From this one can easily show that the triangle is equilateral.

11. Let O be the midpoint of the side AB of a triangle ABC and let the incircle s and the excircle s_1 of that triangle touch AB at points P and Q. We will show that $OP = OQ$. Let a, b, and c be the lengths of the sides of the triangle, and let P, P_1, and P_2, and Q, Q_1, and Q_2 be the points at which s and s_1 touch the sides of the triangle. Then

$$\begin{aligned} AP &= \frac{1}{2}(AP + AP_1) = \frac{1}{2}[(c - BP) + (b - CP_1)] \\ &= \frac{1}{2}[c + b - (BP + CP_1)] = \frac{1}{2}[c + b - (BP_2 + CP_2)] \\ &= \frac{1}{2}(c + b - a), \end{aligned}$$

$$OP = AP - AO = \frac{1}{2}(c + b - a) - \frac{1}{2}c = \frac{1}{2}(b - a);$$

we show in a similar way that

$$BQ = \frac{1}{2}(c + b - a), \qquad OQ = \frac{1}{2}(b - a).$$

We apply the inversion with center O and power $OP^2 = OQ^2$. We claim that this inversion fixes s and s_1. Indeed, let the image of, say, the incircle s be a circle s'. s' intersects the circle Σ of inversion in the same two points as s, and, in view of property C of inversion, is also perpendicular to Σ (for it is easy to see that s is perpendicular to Σ: at their point of intersection these two circles form a 90° angle). From this it follows directly that s' coincides with s; a similar argument shows that s_1 goes over to itself.

Now we will explain what is the image of the nine point circle $\overline{S}$ under our reflection. The sides of the triangle ABC are common tangents to s and s_1. We construct a fourth common tangent to these circles; let D and E, M'

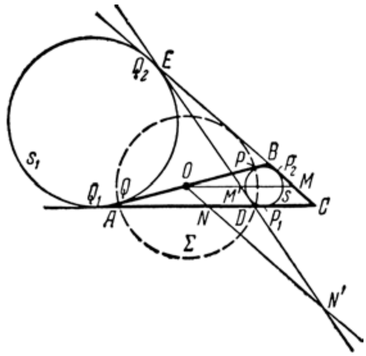


FIGURE 156

and N' be its points of intersection with the sides AC and BC and with the midlines OM and ON respectively. We will prove that *a reflection in* Σ *takes the nine point circle* $\overline{S}$ *to the line* $M'N'$.

Symmetry considerations imply that $CD = CB = a$ and $CE = CA = b$ (see Figure 156). Further, similarity of the triangles $MM'E$ and CDE implies that

$$\frac{MM'}{ME} = \frac{CD}{CE} \quad \text{or} \quad \frac{MM'}{b - \frac{a}{2}} = \frac{a}{b},$$

whence

$$MM' = \frac{a}{b}\left(b - \frac{a}{2}\right),$$

and, therefore,

$$OM' = OM - MM' = \frac{b}{2} - \frac{a}{b}\left(b - \frac{a}{2}\right) = \frac{b^2 + a^2 - 2ab}{2b} = \frac{(b-a)^2}{2b}.$$

In much the same way (using the similarity of the triangles $NN'D$ and CED) we deduce the equality

$$ON' = \frac{(b-a)^2}{2a}.$$

Hence

$$OM' \cdot OM = \frac{(b-a)^2}{2b} \cdot \frac{b}{2} = \frac{(b-a)^2}{4} = OP^2 = OQ^2,$$

and, similarly,

$$ON' \cdot ON = OP^2 = OQ^2.$$

These equalities show that M' and N' are symmetric to M and N with

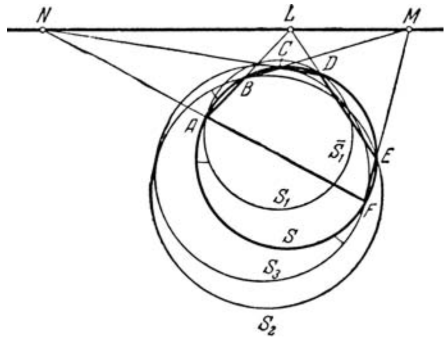


FIGURE 157

respect to Σ. From this it follows that reflection in Σ takes the nine point circle passing through M, N, and O to the line $M'N'$.

The fact that reflection in Σ takes the circles s and s_1 to themselves and the circle $\overline{S}$ to the line $M'N'$ tangent to s and s_1 implies that $\overline{S}$ is tangent to s and s_1.

12. let L, M, and N be the points of intersection of the sides AB and DE, BC and EF, CD and FA of the hexagon $ABCDEF$ inscribed in the circle S (Figure 157). We draw a circle S_1 passing through A and D and apply the inversion with center L and power $k_1 = LB \cdot LA = LD \cdot LE$. This inversion takes A to B, B to A, D to E, and E to D; it takes S to itself and S_1 to a circle S_2 passing through B and E.

Now we apply the inversion with center M and power $k_2 = MC \cdot MB = ME \cdot MF$. This inversion takes S to itself and S_2 to a circle S_3 passing through C and F. Finally, we apply the inversion with center N and power $k_3 = NC \cdot ND = NA \cdot NF$. This inversion takes S to itself and S_3 to a circle $\overline{S}_1$ which intersects S at D and A.

By property C of inversion, the circles S_1, S_2, S_3, and $\overline{S}_1$ form the same angles with S. The fact that S_1 and $\overline{S}_1$, which intersect S at the same points A and D, form the same angles with S implies that these circles coincide.[3]

We see that L, M, and N are the centers of inversions that take S_1 to S_2, S_2 to S_3, and S_3 to S_1. But this means that these points coincide with the centers of similarity of these three circles taken two at a time (see the proof of property B_4 of inversion). In view of the theorem on three centers of similarity (see p. 29 in NML 21), we can now conclude that L, M, and N are collinear.[4]

Our proof remains valid if the hexagon $ABCDEF$ is self-intersecting.

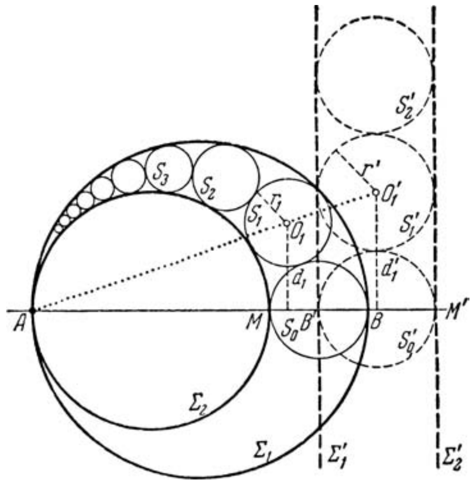


FIGURE 159

13. Apply an inversion that takes the circles Σ_1 and Σ_2 to two parallel lines Σ_1' and Σ_2' (see Theorem 2 on p. 22); its center is the point A of tangency of Σ_1 and Σ_2. The circles $S_0, S_1, S_2, \ldots$ go over to circles $S_0', S_1', S_2', \ldots$ tangent to the parallel lines Σ_1' and Σ_2'; clearly, all of these circles are congruent (Figure 159). Let r' be the radius of each of these circles and let the distances from their centers to AB be $d_0', d_1', d_2', \ldots$. Clearly, $d_n' = 2nr'$, that is, $\frac{r'}{d_n'} = \frac{1}{2n}$. But S_n and S_n' are centrally similar with center of similarity at A (see the proof of property B_4 of inversion). Hence

$$\frac{r_n}{d_n} = \frac{r_n'}{d_n'} = \frac{1}{2n}, \qquad \text{that is,} \qquad d_n = 2nr_n,$$

which was to be proved.

(b) We assume for the sake of simplicity that the power of the inversion used in the solution of problem (a) is $(2R_1)^2$. Then B' in Figure 159 coincides with B and

$$AM' = \frac{(2R_1)^2}{AM} = \frac{4R_1^2}{2R_2} = \frac{2R_1^2}{R_2}.$$

It follows that the common radius r' of $S_0', S_1', S_2', \ldots$ is

$$\frac{1}{2}(AM' - AB') = \frac{1}{2}\left(\frac{2R_1^2}{R_2} - 2R_1\right) = \frac{R_1(R_1 - R_2)}{R_2}.$$

Further, the coefficient of similarity of S_n and S'_n is $\frac{k}{k_n}$, where $k = 4R_1^2$ is the power of the inversion and k_n is the square of the tangent from A to S'_n (see the proof of property B$_4$ of inversion). Clearly, the square of the tangent from A to S'_n is $AO_n'^{\,2} - r'^2$, where O'_n is the center of S'_n, and

$$AO_n'^{\,2} = AO_0'^{\,2} + O'_0O_n'^{\,2}$$

(O'_0 is the center of S'_0, that is, the midpoint of the segment $B'M'$). Hence

$$AO'_0 = 2R_1 + r' = \frac{R_1(R_1+R_2)}{R_2}; \quad O'_0O'_n = d'_n = 2nr',$$

$$AO_n'^{\,2} = \left[\frac{R_1(R_1+R_2)}{R_2}\right]^2 + 4n^2r'^2$$

and the coefficient of similarity of S_n and S'_n is

$$\begin{aligned}\frac{k}{k_n} &= 4R_1^2 : \left\{\left[\frac{R_1(R_1+R_2)}{R_2}\right]^2 + 4n^2r'^2 - r'^2\right\}\\ &= 4R_1^2 : \left\{\left[\frac{R_1(R_1+R_2)}{R_2}\right]^2 + (4n^2-1)\left[\frac{R_1(R_1-R_2)}{R_2}\right]^2\right\}\\ &= 4R_1^2 : \left\{\left[(R_1+R_2)^2 + (4n^2-1)(R_1-R_2)^2\right]\left(\frac{R_1}{R_2}\right)^2\right\}.\end{aligned}$$

It follows that the required radius r_n is

$$r_n = \frac{k}{k_n}r' = \frac{4R_1R_2(R_1-R_2)}{(R_1+R_2)^2 + (4n^2-1)(R_1-R_2)^2}.$$

14. We apply an inversion that takes Σ_1 and Σ_2 to two intersecting lines Σ'_1 and Σ'_2; its center is the point A of intersection of Σ_1 and Σ_2. The inversion takes the circles $S_1, S_2, S_3, \ldots$ to circles $S'_1, S'_2, S'_3, \ldots$ inscribed in the angle formed by Σ'_1 and Σ'_2 (Figure 160); clearly, if $r'_1, r'_2, r'_3, \ldots$ are the radii of the circles $S'_1, S'_2, S'_3, \ldots$ and $d'_1, d'_2, d'_3, \ldots$ are the distances from their centers to the line AB, then

$$\frac{r'_1}{d'_1} = \frac{r'_2}{d'_2} = \frac{r'_3}{d'_3} = \cdots$$

(the circles S'_1, S'_2, S'_3 are centrally similar with center of similarity B'). But the circles $S_1, S_2, S_3, \ldots$ are centrally similar to the circles S'_1, S'_2, S'_3 with

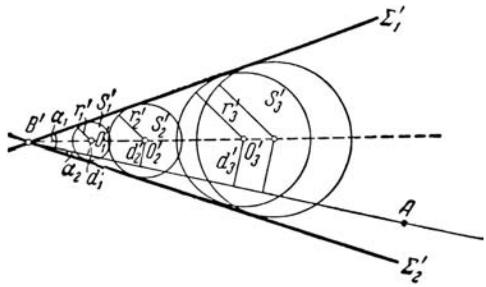


FIGURE 160

center of similarity A; hence

$$\frac{r_1}{d_1} = \frac{r_1'}{d_1'}, \quad \frac{r_2}{d_2} = \frac{r_2'}{d_2'}, \quad \frac{r_3}{d_3} = \frac{r_3'}{d_3'},$$

which implies that

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \frac{r_3}{d_3} = \dots$$

Further, using the designations of Figure 160, we have

$$\frac{r_1'}{d_1'} = \frac{r_1'}{O_1'B'} : \frac{d_1'}{O_1'B'} = \frac{\sin\angle O_1'B'P}{\sin\angle O_1'B'Q}$$

$$= \frac{\sin\frac{\alpha_1+\alpha_2}{2}}{\sin(\frac{\alpha_1+\alpha_2}{2} - \alpha_2)} = \frac{\sin\frac{\alpha_1+\alpha_2}{2}}{\sin\frac{\alpha_1-\alpha_2}{2}}.$$

Hence

$$\sigma = \frac{\sin\frac{\alpha_1+\alpha_2}{2}}{\sin\frac{\alpha_1-\alpha_2}{2}},$$

where, by property C of inversion, α_1 and α_2 are the angles formed by Σ_1 and Σ_2 with the common chord AB.

15. (a) We apply an inversion that takes Σ and the line AB to two intersecting lines Σ' and AB (its center is the endpoint A of the diameter AB of Σ); by property C of inversion, Σ' is perpendicular to AB. We take the power of the inversion to be $2R$; then B is fixed. The circles $S_0, S_1, S_2, \dots$ go over to circles $S_0', S_1', S_2', \dots$ in the right angle formed by AB and Σ' (Figure 161).

The center of Σ goes over to a point O' such that

$$AO' = \frac{AB^2}{AO} = \frac{4R^2}{R} = 4R;$$

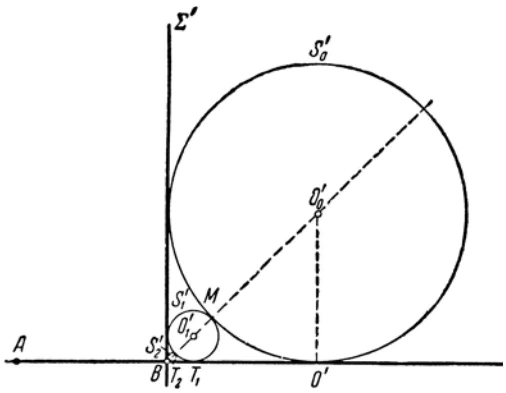


FIGURE 161

hence the radius $r'_0 = BO'$ of S'_0 is $4R - 2R = 2R$. Further, the pair of circles S'_2, S'_1 is centrally similar to the pair of circles S'_1, S'_0 with center of similarity B; hence

$$\frac{r'_2}{r'_1} = \frac{r'_1}{r'_0}$$

(here $r'_1, r'_2, \ldots$ are the radii of the circles $S'_1, S'_2, \ldots$). Put $r'_1/r'_0 = \omega$. Then

$$r'_1 = r'_0\omega, \qquad r'_2 = r'_1\omega = r'_0\omega^2.$$

In much the same way we show that

$$r'_3 = r'_2\omega = r'_0\omega^3, \qquad r'_4 = r'_0\omega^4, \ldots, r'_n = r'_0\omega^n.$$

It is easy to determine ω. Let O'_0 and O'_1 be the centers of S'_0 and S'_1 and let M be their point of tangency. Clearly,

$$O'_0B = r'_0\sqrt{2}, \qquad O'_1B = O'_0B - (r'_0 + r'_1) = r'_0(\sqrt{2} - 1) - r'_1$$

and

$$O'_1B = r'_1\sqrt{2}.$$

Hence

$$r'_0(\sqrt{2} - 1) - r'_1 = r'_1\sqrt{2}, \qquad \omega = \frac{r'_1}{r'_0} = \frac{\sqrt{2} - 1}{\sqrt{2} + 1},$$

and

$$r'_n = r'_0\omega^n = 2R\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^n \qquad (n = 0, 1, 2, \ldots).$$

Now we make use of the fact that the circles S_n and S'_n are centrally similar with coefficient of similarity $\frac{k}{k_n}$, where $k = 4R^2$ is the power of the inversion and k_n is the square of the tangent from A to S'_n. Let $T_1, T_2, \ldots$ be the points at which $S_1, S_2, \ldots$ touch AB; considerations of similarity imply that

$$\frac{BT_n}{BO'} = \frac{r'_n}{r'_0} = \omega^n,$$

hence

$$BT_n = BO' \cdot \omega^n = 2R\omega^n, \qquad AT_n = 2R + 2R \cdot \omega^n = 2R(1 + \omega^n).$$

Therefore

$$\frac{k}{k_n} = \frac{4R^2}{4R^2(1+\omega^n)^2} = \frac{1}{(1+\omega^n)^2}.$$

All these yield for r_n the value

$$r_n = \frac{k}{k_n} r'_n = \frac{1}{(1+\omega^n)^2} 2R\omega^n = 2R\frac{\omega^n}{(1+\omega^n)^2},$$

where $\omega = (\sqrt{2}-1)/(\sqrt{2}+1)$.

(b) Using the result in (a), we have

$$2t_n = \frac{2R}{r_n} = \frac{(1+\omega^n)^2}{\omega^n} = \frac{1 + 2\omega^n + \omega^{2n}}{\omega^n} = \omega^n + \frac{1}{\omega^n} + 2,$$

where

$$\omega = \frac{\sqrt{2}-1}{\sqrt{2}+1}, \qquad \frac{1}{\omega} = \frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Eliminating the radicals in the denominators of the fractions ω and $\frac{1}{\omega}$ we get:

$$\omega = \frac{(\sqrt{2}-1)^2}{2-1} = 3 - 2\sqrt{2}; \qquad \frac{1}{\omega} = \frac{(\sqrt{2}+1)^2}{2-1} = 3 + 2\sqrt{2};$$

hence

$$2t_n = (3 - 2\sqrt{2})^n + (3 + 2\sqrt{2})^n + 2.$$

[This formula implies readily that t_n is a whole number for all values of n.]

The sum $\omega + \frac{1}{\omega}$ is a whole number: $\omega + \frac{1}{\omega} = 6$. Since the product $\omega \cdot \frac{1}{\omega} = 1$, ω and $\frac{1}{\omega}$ are the roots of the quadratic equation

$$x^2 - 6x + 1 = 0.$$

Now we prove that

$$t_n = 6t_{n-1} - t_{n-2} - 4.$$

Clearly,

$$\begin{aligned}
&2t_n - 6 \cdot 2t_{n-1} + 2t_{n-2} + 8 \\
&\quad = \left[\omega^n + \left(\frac{1}{\omega}\right)^n + 2\right] - 6\left[\omega^{n-1} + \left(\frac{1}{\omega}\right)^{n-1} + 2\right] \\
&\qquad + \left[\omega^{n-2} + \left(\frac{1}{\omega}\right)^{n-2} + 2\right] + 8 \\
&\quad = \omega^{n-2}(\omega^2 - 6\omega + 1) + \left(\frac{1}{\omega}\right)^{n-2}\left[\left(\frac{1}{\omega}\right)^2 - 6\left(\frac{1}{\omega}\right) + 1\right] = 0,
\end{aligned}$$

for $\omega^2 - 6\omega + 1 = \left(\frac{1}{\omega}\right)^2 - 6\left(\frac{1}{\omega}\right) + 1 = 0$. This implies the required formula.

To complete the solution of the problem it suffices to note that

$$\begin{aligned}
2t_0 &= (3 - 2\sqrt{2})^0 + (3 + 2\sqrt{2})^0 + 2 = 4; \\
2t_1 &= (3 - 2\sqrt{2}) + (3 + 2\sqrt{2}) + 2 = 8.
\end{aligned}$$

16. The statement of this problem is rather involved but its solution is relatively straightforward. The definition of a chain implies that inversion takes a chain of circles to a chain of circles. But any pair of nonintersecting circles can be taken by inversion to a pair of concentric circles (see Theorem 2 on p. 22). Hence every chain can be changed by inversion to a simpler chain based on a pair of concentric circles (Figure 162). This remark implies all the assertions of the problem.

(a) The relevant assertion is obvious for a pair of concentric circles Σ_1', Σ_2' and must therefore hold for any pair of nonintersecting circles.

(b) Clearly, for a pair of concentric circles Σ_1', Σ_2' to serve as a base of a chain of circles it is necessary and sufficient that the angle α, between the tangents l_1', l_2' from the common center O of Σ_1' and Σ_2' to any circle S' tangent to Σ_1' and Σ_2' (to one externally and to the other internally; see Figure 162), be commensurable with 360°.

If $\alpha = \frac{m}{n} \cdot 360°$, then, clearly, the chain with base Σ_1', Σ_2' contains n circles, and the points of tangency between the circles of the chain and Σ_1', taken in the same order as that of the circles of the chain, traverse Σ_1' m

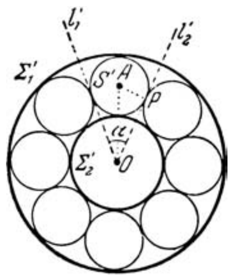


FIGURE 162

times. Going back by inversion from the pair Σ_1', Σ_2' to any pair Σ_1, Σ_2 of nonintersecting circles we see that it can serve as a base of a chain if and only if the angle α between two circles l_1 and l_2, perpendicular to Σ_1 and Σ_2 and tangent to a circle S—which, in turn, is tangent to Σ_1 and Σ_2 (in the way required of the circles of the chain, that is, in the same way if Σ_1 and Σ_2 are exterior to one another, and in opposite ways if the smaller one of these circles is in the interior of the larger one)—is commensurable with 360°; an extra insight is that this angle depends only on Σ_1 and Σ_2 and not on the choice of S. However, one can obtain a more explicit expression for α.

We return to the case of a pair of concentric circles Σ_1' and Σ_2'. Let $S^{1'}$, $S^{2'}$ be a pair of circles tangent to Σ_1' and Σ_2' as shown in Figure 163. We will prove that the angle between $S^{1'}$, $S^{2'}$ is α. Let the radii of Σ_1' and Σ_2' be r_1 and r_2; assume for definiteness that $r_1 > r_2$. Clearly, the radius of a circle S' tangent to Σ_1' and Σ_2' is $\frac{r_1-r_2}{2}$, and the radii of the circles $S^{1'}$ and $S^{2'}$ are $\frac{r_1+r_2}{2}$; the distance from O to the center of S' is $\frac{r_1+r_2}{2}$, and the distances from O to the centers of $S^{1'}$ and $S^{2'}$ are $\frac{r_1-r_2}{2}$. From this we

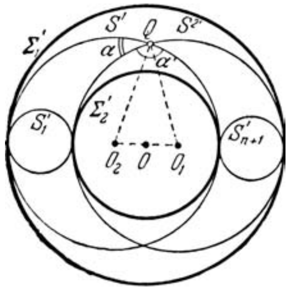


FIGURE 163

readily deduce that $\alpha = 2\arcsin\frac{r_1-r_2}{r_1+r_2}$ (in the right triangle in Figure 162

$$OA = \frac{r_1+r_2}{2}, \quad AP = \frac{r_1-r_2}{2}\bigg).$$

By joining the centers of $S^{1'}$ and $S^{2'}$ to a point Q of their intersection we obtain an isosceles triangle with base $2\frac{r_1-r_2}{2}$ and lateral side $\frac{r_1+r_2}{2}$; this implies that the angle between the radii of $S^{1'}$ and $S^{2'}$ that terminate at a point of their intersection, or—which is the same thing—the angle between $S^{1'}$ and $S^{2'}$, is also $2\arcsin\frac{r_1-r_2}{r_1+r_2}$, which is what we were to prove.

Now using an inversion to go from the pair Σ_1', Σ_2' to a pair of nonintersecting circles Σ_1 and Σ_2, we find that the angle α is equal to the angle between a pair of circles S^1, S^2 tangent to Σ_1 and Σ_2 at the points of intersection of Σ_1 and Σ_2 with a circle perpendicular to Σ_1 and to Σ_2 (the common diameter of Σ_1 and Σ_2 goes over to this circle). Also, S^1 and S^2 are not tangent to Σ_1 and Σ_2 in the way in which circles of a chain with base Σ_1 and Σ_2 are tangent to these circles (that is, S^1 and S^2 touch Σ_1 and Σ_2 the same way if Σ_1 and Σ_2 are inside one another and in different ways if they are outside one another). This, in particular, implies the proposition in 16(b).

(c) We again consider a pair of concentric circles Σ_1', Σ_2'. Let S_1' and S_{n+1}' be circles tangent to Σ_1' and Σ_2' in diametrically opposite points (in the way in which circles of a chain are supposed to touch Σ_1' and Σ_2'; see Figure 163).

It is clear that the circles $S^{1'}$ and $S^{2'}$ that appear in problem 16(b) are the same for the pairs Σ_1', Σ_2' and S_1', S_{n+1}'. We leave it to the reader to try to explain why one should not regard the angles α and α' as equal but as supplementary ($\alpha + \alpha' = 180° = \frac{1}{2}\cdot 360°$).

Notes to Section 1

[1] Four circles every three of which meet in a point need not all meet in a point (four such circles can be obtained by applying an inversion to the sides of a triangle and its circumcircle). But if n ($\geqq 5$) pairwise different circles are such that every three of them meet in a point, then all n of them must meet in a point.

[2] Our solution of problem 7(a) is based on the theorems in problems 6(a) and 6(b). One can also solve this problem by using the theorem in problem 6(c).

[3] To claim with certainty that $\overline{S}_1$ coincides with S_1 we must show that the angles formed by these circles with S are not only equal but also directed in the same way (the circles S_1 and S_1' in Figure 158 form equal angles with S and intersect it in the same points but nevertheless do not coincide).

It is easy to show that this condition is satisfied. In fact, assume that the angle between S and S_1 at A is α and is directed counterclockwise (that is, the tangent to S at A can be made to coincide with the tangent to S_1 by rotating it counterclockwise through α); in that case, clearly, the angle between S and S_1 at D is also equal to α but is directed clockwise (see, for example, Figure 158). Further, in view of the properties of inversion (see the remark on p. 13), the angle between S and S_2 at B is equal to α and is directed clockwise, the angle between S and S_3 at C is equal to α and is directed counterclockwise, and, finally, the angle between S and $\overline{S}_1$ at D is equal to α and is directed counterclockwise. Thus S_1 and $\overline{S}_1$ form with S angles that coincide in magnitude as well as in direction.

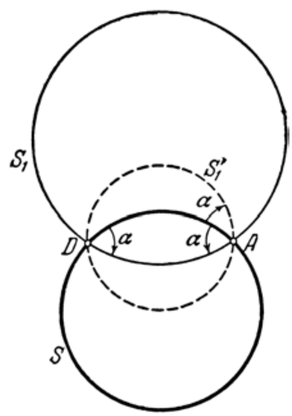


FIGURE 158

[4] We leave it to the reader to prove that the three centers of similarity of the circles S_1, S_2, and S_3 taken two at a time lie on a line.

Section 2

17. Assume that the problem has been solved. The equality $OX \cdot OY = k$ implies that X is obtained from Y on the line AN by applying an inversion with center O and power k; therefore X is on the circle S obtained from AN by the inversion with center O and power k (or $-k$), that is, X is the point of intersection of the line AM and the circle S (which can be constructed). The problem can have up to four solutions.

18. Assume that the problem has been solved and $MNPQ$ is the required parallelogram. The center of $MNPQ$ coincides with the center of the given parallelogram $ABCD$ (Figure 164; see the solution of problem 30(b) on p. 38 in NML 21). We are given the angle $MON = \alpha$ and we know that $2OM \cdot ON \sin\alpha = S$ (S is the area of $MNPQ$). If N' is obtained from N by a rotation through α about O, then $ON'M$ is a single line and

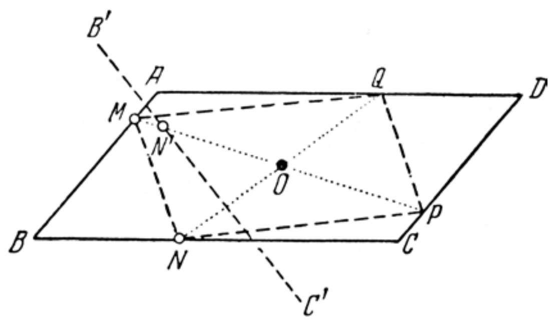


FIGURE 164

$OM \cdot ON' = \frac{S}{2\sin\alpha} = k$ is a known quantity. Since N' is on the line $B'C'$ obtained from BC by the rotation through α about O, we are back to the preceding problem: draw a line through O which intersects given lines AB and $B'C'$ in points M and N' such that $OM \cdot ON' = k$.

19. (a) Assume that the line l has been drawn and let X and Y be the feet of the perpendiculars dropped from B and C to l; $BX \cdot CY = k$ (Figure 165). We move the triangle AYC parallel to the segment CB through a distance CB to the position $A'Y'B$; A' is easily determined from A, B, and C. Since $A'Y'B$ is a right angle, the point Y' is on the circle S with diameter $A'B$; since $BX \cdot BY' = BX \cdot CY = k$, X is obtained from Y' by the inversion with center B and power k (or $-k$). Hence X is on the line S'

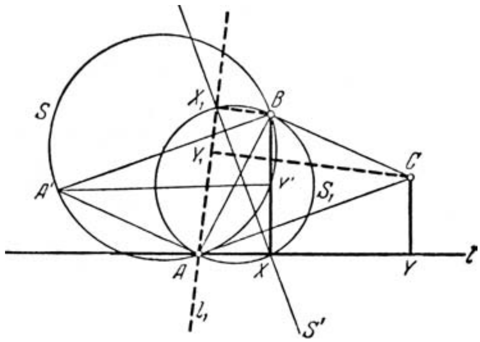


FIGURE 165

obtained from S by this inversion. Also, X is on the circle S_1 with diameter AB (for $\angle AXB = 90°$); hence X is the point of intersection of the line S' and the circle S_1. The problem can have up to four solutions.

(b) Let X and Y be the feet of the perpendiculars from B and C to the required line l, $CY^2 - BX^2 = k_1$ (Figure 166). We lay off on the line BX, on both sides of X, segments XD and XD' equal to YC; then

$$BD \cdot BD' = (CY - BX)(CY + BX) = k_1.$$

Let C_1 be a point symmetric to C with respect to A; then $C_1D' \parallel CD \parallel l$. We apply the inversion with center B and power k_1 (or $-k_1$); let C' be the image of C_1 under this inversion. The line C_1D' goes over to the circle S passing through B, C', and D; since $C_1D' \perp BD'$, the segment BD is a

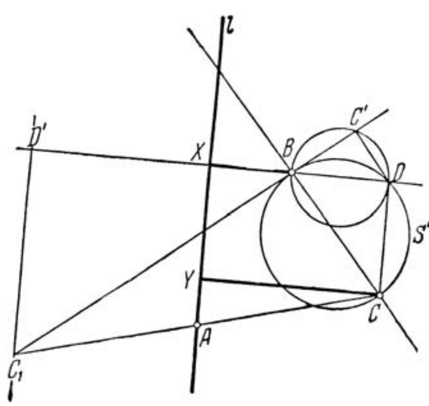


FIGURE 166

diameter of S. It follows that D is on the perpendicular to BC_1 erected at C'. Since, in addition, $\angle BDC = 90°$ (for $CD \parallel l$), D is also a point of the circle S_1 with diameter BC. The problem can have two solutions.

20. Let $A_1A_2A_3 \ldots A_n$ be the required polygon whose sides A_1A_2, A_2A_3, $\ldots$, $A_{n-1}A_n$, A_nA_1 pass through the given points $M_1, M_2, \ldots, M_{n-1}, M_n$ (Figure 167).

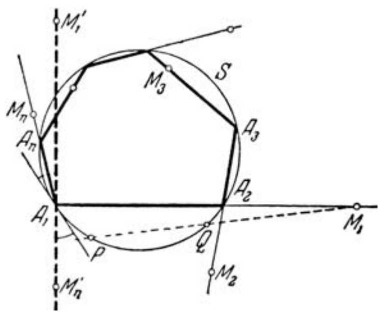


FIGURE 167

We apply the inversion I_1 with center M_1 and power $\pm\ M_1A_1 \cdot M_1A_2$ (the plus sign is used if M_1 is outside S and the minus sign if M_1 is inside S); the power of this inversion is easy to determine by passing through M_1 a secant PQ of the circle S and keeping in mind the fact that $M_1A_1 \cdot M_1A_2 = M_1P \cdot M_1Q$. This inversion takes S to itself and the vertex A_1 of the required polygon to the vertex A_2. Next we apply the inversion I_2 with center M_2 and power $\pm\ M_2A_2 \cdot M_2A_3$, then the inversion I_3 with center M_3 and power $\pm\ M_3A_3 \cdot M_3A_4$, and so on, ending with the inversion I_n with center M_n and power $\pm M_nA_n \cdot M_nA_1$. The inversions $I_1, I_2, \ldots, I_n$ take S to itself; they take A_1 successively to $A_2, A_3, A_4, \ldots$ and finally back to A_1.

Now let S_1 be a line or a circle passing through A_1. The succession of inversions $I_1, I_2, \ldots, I_n$ takes S_1 to a circle (or line) S_1' also passing through A_1; knowing S_1' we can determine the initial circle or line S_1 (for S_1 can be obtained from S_1' by applying to it the inversions $I_n, I_{n-1}, \ldots, I_2, I_1$ in the indicated order: first I_n, then I_{n-1}, and so on).

It is clear that if the circle (or line) S_1 is to go over to the line S_1', then the circle or line S_n, obtained from S_1 as a result of the application of the succession of inversions $I_1, I_2, \ldots, I_{n-1}$, must pass through the center M_n of the last of these inversions, or, equivalently, S_1 must pass through the

point M'_n which goes over to M_n as a result of the succession of inversions $I_1, I_2, \dots, I_{n-1}$; conversely, if S_1 passes through M'_n, then S'_1 is a line. It is easy to obtain M'_n—it is the image of M_n under the succession of inversions $I_{n-1}, \dots, I_2, I_1$ applied in the indicated order. Similarly, if S_1 is a line, then the circle (or line) S'_1 passes through the point M'_n, which is the image of M_1 under the succession of inversions $I_2, I_3, \dots, I_n$ (for I_1 takes the line S_1 to the circle S_2 passing through M_1).

Now consider the line $A_1M'_n$. Since it passes through M'_n, it goes over under the n successive inversions to a line; since $A_1M'_n$ is a line, the resulting line passes through M'_1. But the succession of n inversions takes A_1 to itself. Hence *the sequence of n inversions $I_1, I_2, \dots, I_n$ takes the line $A_1M'_n$ to the line $A_1M'_1$*. At this point we must distinguish two cases.

1°. n is even (Figure 167). In that case, application of the sequence of n inversions changes neither the size nor the direction of the angle between the line M'_nA_1 and the circle S (see property C of inversion on p. 11 and the remark on p. 12). It follows that the lines M'_nA_1 and M'_1A_1, which form at A_1 equal angles with the circle S, coincide. The point A_1 can be obtained as the point of intersection of the line $M'_nM'_1$ with S. Depending on the disposition of $M'_nM'_1$ and S the problem can have two solutions, one solution, or no solution; in the exceptional case of coincidence of M'_n and M'_1 the problem is undetermined (in that case we can take as the vertex A_1 of the required polygon any point of the circle S).

2°. n is odd. In that case the lines M'_nA_1 and M'_1A_1 form with S angles equal in magnitude but oppositely directed. Thus our problem reduces to finding on S a point A_1 such that the lines M'_nA_1 and M'_1A_1 form with S angles equal in magnitude but oppositely directed (that is, such that M'_nA_1 and M'_1A_1 are symmetric with respect to the radius OA_1 of S). This turns out to be a very complicated problem. This being so, we use a trick that reduces the case of an odd n to the case of an even n. We consider the sequence of $n+1$ inversions $I_1, I_2, I_3, \dots, I_n, I$ where I is a reflection in S. This sequence of $n+1$ inversions takes S to itself; A_1 to A_1; any circle passing through A_1 and the point O', which is the image of the center O of S under the sequence of n inversions $I_n, I_{n-1}, I_{n-2}, \dots, I_1$, to a line passing through A_1; and any line passing through A_1 to a circle or line passing through A_1 and the image M' of M_1 under the sequence of inversions $I_2, I_3, \dots, I_n, I$. Since $n+1$ is even, it follows as before that A_1 is a point common to S and the line $M'O'$. Depending on the number of common points of S and $M'O'$, the problem has two solutions, one solution, or no solution. If M' and O' coincide, then the problem is

undetermined.

If the directions of some of the sides of the polygon are known (but not the points through which these sides pass), then the corresponding inversions are replaced by reflections in the diameters of S perpendicular to the given directions. Also, if in the sequence $I_1, I_2, I_3, \ldots, I_n, I$ of inversions or reflections in lines the first k transformations $I_1, I_2, \ldots, I_k$ and the last $n-l$ transformations $I_{l+1}, I_{l+2}, \ldots, I_n$ are reflections in lines and I_{k+1} and I_l are inversions (here $k+1$ can be 1 and l can be n; if n is odd, then the sequence ends with the inversion $I_n = I$ and I_l coincides with that inversion), then the role of the point M'_n is played by the point M'_l, the image of the center of the inversion I_l under the sequence $I_{l-1}, I_{l-2}, \ldots, I_1$ of inversions or reflections in lines, and the role of the point M'_1 is played by the point M'_{k+1}, the image of the center M_{k+1} of the inversion I_{k+1} under the sequence $I_{k+2}, I_{k+3}, \ldots, I_n$ (I) of inversions or reflections in lines.

Remark 1. It is possible to solve the problem for an odd n without introducing the extra inversion I. In fact, one can show that *the points M'_1 and N'_1 must be equidistant from the center O of S*; this special property makes it easy to find the point A_1. For proof (of this property) note that if n is odd, then the sequence $I_1, I_2, \ldots, I_n$ of n inversions takes the line $M'_1M'_n$ not to itself but to a line passing through M'_1 (for $M'_1M'_n$ is a line passing through M'_n) and takes to it a line b passing through M'_n. By property C of inversion, the angle between b and $M'_1M'_n$ is equal to the angle between $M'_1M'_n$ and a; on the other hand, by the same property, all three lines form equal angles with S (we are assuming that they intersect S). This implies that M'_1 and N'_1 are equidistant from O.

Remark 2. The idea of the present solution is very close to the idea of the first solution of problem 15 on p. 28 in NML 8. One could also present a solution of problem 20, whose idea is close to that of the second solution of problem 15 on p. 87 in NML 8. This is a more substantive solution than the one just given. The reason we did not do this is that it calls for the development of the fairly complicated theory of composition of inversions (the product of two inversions is not an inversion), a theory of which we make no use in the sequel.

21. (a) Apply an inversion with center A. Then B goes over to a point B', the circle (or line) S to a circle (or line) S', and the required circle Σ to a circle Σ' which passes through B' and is tangent to the circle S' (or parallel to the line S'). After constructing Σ' we can easily obtain the circle (or line) Σ. The problem has two solutions, one solution, or no solution.

(b) The solution of this problem is very similar to the solution of problem (a). We apply an arbitrary inversion with center A. Then S_1 and S_2 go over to new circles S'_1 and S'_2 and the required circle Σ goes over to a common tangent Σ' of S'_1 and S'_2. The latter can be constructed.

The problem has up to four solutions (see Figure 168).

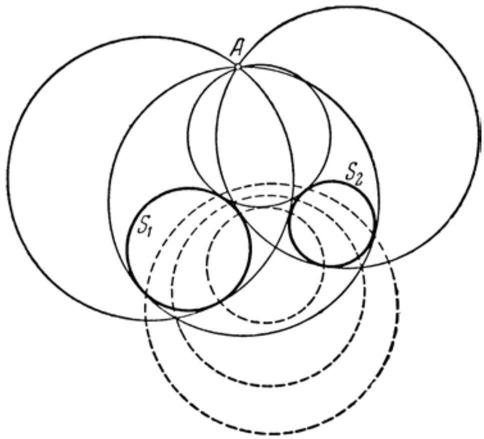


FIGURE 168

Remark. If S_1' and S_2' have four common tangents, then symmetry considerations imply that the points of tangency of S_1' and S_2' with the common outer tangents are concyclic, the points of tangency of S_1' and S_2' with the common inner tangents are concyclic, and the points of intersection of the common outer tangents with the common inner tangents are also concyclic (it is not difficult to show that these three circles are concentric and their common center is the midpoint of the segment $O_1'O_2'$, where O_1' and O_2' are the centers of S_1' and S_2'). By property B_4 of inversion, this implies that the eight points of tangency of S_1 and S_2 with the four circles passing through A and tangent to S_1 and S_2 (here we are assuming that there are four such circles) lie in sets of four on two circles, and that the points of intersection other than A_1 of the circles tangent to S_1 and S_2 in a like-name manner with the circles tangent to S_1 and S_2 in an unlike-name manner are concyclic (Figure 168; we note that since inversion does not take the center of a circle to the center of the image circle, these three circles are not concentric).

22. (a) In accordance with the definition of a reflection in a circle (see pp. 1–4), the required circle (or line) Σ must pass also through the point A' symmetric to A with respect to the circle (or line) S. If $A' \neq B$, then the problem has a unique solution; if $A' = B$ the problem is undetermined.

Second solution. Under an inversion with center A, the point B goes over to a new point B', the circle S goes over to a circle (or line) S', and the required circle (or line) Σ goes over to a line Σ' passing through B' and perpendicular to the circle (or line) S' (that is, passing through the center of S' if the latter is a circle). After constructing Σ' we can easily find Σ. In general, the problem has a unique solution; if B' coincides with the center of S', then the problem is undetermined.

(b) Let O be the center of S and r its radius. The required circle (or line) Σ must pass through the point A', obtained from A by the inversion with center O and power $-r^2$ (see the text in fine print on p. 6). In general, the problem has a unique solution; if A' coincides with B the problem is undetermined.

23. (a) In accordance with the definition of a reflection in a circle, the required circle (or line) Σ must pass through the points A' and A'', symmetric to A with respect to the circles (or lines) S_1 and S_2. In general, the problem has a unique solution (if A' coincides with A'', then the problem is undetermined, but if A is one of the two points of intersection of S_1 and S_2, then the problem has no solution).

Second solution. An inversion with center A takes the circles S_1 and S_2 to circles (or a circle and a line, or two lines) S_1' and S_2', and the required circle (or line) Σ to the line Σ' perpendicular to S_1' and S_2' (that is, passing through the centers of S_1' and S_2' if the latter are circles). After constructing Σ' we can easily find Σ. In general, the problem has a unique solution; if S_1' and S_2' are concentric circles or parallel lines, then the problem is undetermined; if S_1' and S_2' are intersecting lines, then the problem has no solution.

(b) Let O_1 and O_2 be the centers, and r_1 and r_2 the radii, of the circles S_1 and S_2. The required circle (or line) Σ must pass through the point A', obtained from A by the inversion with center O_1 and power r_1^2 (if S_1 is a line, then A' is symmetric to A with respect to S_1), and through the point A'', obtained from A by the inversion with center O_2 and power $-r_2^2$ (see the solutions of problems 22(a) and 22(b)). In general, the problem has a unique solution; if A' coincides with A'', then the problem is undetermined.

(c) Let O_1 and O_2 be the centers, and r_1 and r_2 the radii, of the circles S_1 and S_2. The required circle (or line) Σ must pass through the points A' and A'', obtained from A by the inversions with centers O_1 and O_2 and powers $-r_1^2$ and $-r_2^2$. In general, the problem has a unique solution; if A' coincides with A'', then the problem is undetermined.

24. (a) An inversion with center A takes the point B to another point B' and the circle (or line) S to a circle (or line) S'; the required circle (or line) Σ goes over to a line Σ' which cuts S' at a given angle α. But if Σ' cuts the known circle S' at a given angle α (Figure 169), then the arc cut off by Σ' on the circle $\overline{S}'$ is 2α, and, consequently, we know the distance from Σ' to the center of S'; in other words, Σ' is tangent to $\overline{S}'$, concentric with

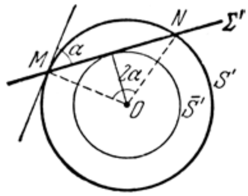


FIGURE 169

S', which we can construct. After constructing Σ' we can obtain Σ. The problem can have two solutions, one solution, or no solution.

(b) An inversion with center A takes S_1 and S_2 to circles (or lines, or a circle and a line) S_1' and S_2', and the required circle (or line) Σ to a line Σ' which cuts S_1' and S_2' at angles α and β, that is,—if S_1' and S_2' are circles—to a common tangent of the auxiliary circles $\overline{S}_1'$ and $\overline{S}_2'$ (see the solution of problem (a)). After constructing Σ' we can obtain Σ. The problem can have up to four solutions; if S_1' and S_2' are lines, then the problem has no solution or is undetermined.

25. We consider separately three cases.

1°. S_1 and S_2 intersect one another. We take S_1 and S_2 by an inversion to two intersecting lines S_1' and S_2' (see Theorem 2 in Section 1, p. 22; the solution of the problem is simplified if S_1 and S_2 are lines to begin with). This inversion takes the required circle (or line) Σ to a circle Σ' with center at the point O of intersection of S_1' and S_2' (for the center of a circle perpendicular to S_1' and S_2' must lie on both lines). If we can construct Σ' then we can easily construct Σ.

2°. S_1 and S_2 are tangent to one another. In that case we can take S_1 and S_2 by an inversion to two parallel lines S_1' and S_2' (see Theorem 2 in Section 1, p. 22; the solution of the problem is simplified if S_1 and S_2 are parallel lines to begin with). The required circle (or line) Σ goes over to a (circle or) line Σ' perpendicular to S_1' and S_2'; clearly, Σ' is a line perpendicular to the common direction of S_1' and S_2'. If we can construct Σ' then we can easily construct Σ.

3°. S_1 and S_2 are disjoint. In that case we can take S_1 and S_2 by an inversion to two concentric circles S_1' and S_2' (see Theorem 2 in Section 1, p. 22; the solution of the problem is simplified if S_1 and S_2 are concentric circles to begin with). The required circle (or line) Σ goes over to a (circle or) line Σ' perpendicular to S_1' and S_2', that is, to a line passing through

the common center O of the circles S_1' and S_2' (see pp. 45–46). If we can construct Σ' then we can easily construct Σ.

We analyze separately problems (a), (b), and (c).

(a) In the case 1° the problem is reduced to constructing a circle Σ' with given center O, perpendicular to the circle or line S_3', which is the image of S_3 under our inversion (if S_3' is a circle, then the radius of Σ' is equal to the segment of the tangent from O to S_3'); in the case 2° the problem is reduced to constructing a line Σ' with given direction, perpendicular to the circle (or line) S_3' (if S_3' is a circle, then Σ' passes through its center); in the case 3° the problem is reduced to constructing a line Σ' passing through the given point O and perpendicular to the circle (or line) S_3' (if S_3' is a circle, then Σ' passes through its center). In general, the problem has a unique solution; in the case 1° the problem has no solution if O is in the interior of S_3'; in the case 3° the problem is undetermined if O is the center of S_3'; if S_1, S_2, and S_3 are three lines, then the problem has no solution or is undetermined.

(b) In the case 1° the problem is reduced to constructing a circle Σ' with given center O, tangent to the circle (or line) S_3' which is the image of S_3 under our inversion; in the case 2° the problem is reduced to constructing a line Σ' with given direction tangent to the circle S_3' (or parallel to the line S_3'); in the case 3° the problem is reduced to constructing a line Σ' passing through the given point O and tangent to the circle S_3' (or parallel to the line S_3'). The problem has two solutions, one solution, or no solution; if S_1, S_2, and S_3 are three lines, then the problem has no solution or is undetermined.

(c) In the case 1° the problem is reduced to constructing a circle Σ' with given center O that cuts the circle (or line) S_3', the image of S_3 under our inversion, at an angle α. If S_3' is a circle with center O_3' which intersects Σ' at A, then we know the angle OAO_3', namely, $\angle OAO_3' = \alpha$ (see p. 2); this enables us to determine A and therefore also Σ'. If S_3' is a line that intersects Σ' in points A and B, then we know the angles of the isosceles triangle AOB; this enables us to determine Σ'.

In the case 2°, the problem is reduced to constructing a line Σ' with given direction that cuts the circle (or line) S_3' at an angle α (if S_3' is a circle, then Σ' is tangent to a determined circle $\overline{S}_3'$ concentric with S_3'; see the solution of problem 24(a)).

In the case 3°, the problem is reduced to constructing a line Σ' passing through a given point O and cutting a known circle (or line) S_3' at an angle α (if S_3' is a circle, then the line Σ' is tangent to a determined circle $\overline{S}_3'$).

The problem has two solutions, one solution, or no solution; if S_1' and S_2' are parallel lines, then the solution of the problem can also be undetermined.

26. The problem can be solved in a manner similar to the solution of problem 25. We consider separately three cases.

1°. S_1 and S_2 intersect one another. In this case one can take S_1 and S_2 by inversion to intersecting lines S_1' and S_2'; this inversion takes the required circle (or line) Σ to a circle Σ' tangent to these lines.

2°. S_1 and S_2 are tangent to one another. In this case one can take S_1 and S_2 by inversion to two parallel lines S_1' and S_2'; this inversion takes the required circle (or line) Σ to a circle Σ' tangent to S_1' and S_2', or to a line Σ' parallel to S_1' and S_2' (if Σ' is a circle, then its center is on the midline of the strip formed by S_1' and S_2', and its radius is equal to half the distance between S_1' and S_2').

3°. S_1 and S_2 have no common points. In this case one can take S_1 and S_2 by an inversion to two concentric circles S_1' and S_2'; this inversion takes the required circle (or line) Σ to a circle Σ' tangent to S_1' and S_2' (the center of Σ' lies on the circle concentric with S_1' and S_2' whose radius is half the sum or difference of the radii of S_1' and S_2', and the radius of Σ' is, respectively, half the difference or sum of the radii of S_1' and S_2'; Figure 170).

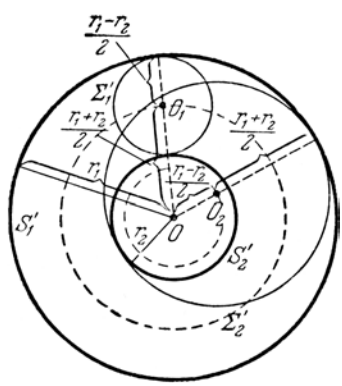


FIGURE 170

Now we consider our problems (a) and (b).

(a) *First solution.* In cases 1° and 2° our problem is reduced to the construction of a circle Σ' tangent to the lines S_1' and S_2' and to the known

circle (or line) S_3'—the image of S_3 under our inversion (if S_3' is a circle, then the problem is identical with problem 13(c) on p. 18 in NML 21). In case 3° the problem is reduced to the construction of a circle Σ' tangent to the concentric circles S_1' and S_2' and the known circle (or line) S_3'. This problem is easily solved, for we know the radius of Σ' and the circle one of whose points is the center of Σ' (see Figure 170).

The problem can have up to eight solutions; if S_1', S_2', and S_3' are parallel lines (that is, S_1', S_2', and S_3' are tangent at a single point), then the problem is undetermined.

Another solution of this problem is given after the solution of problem (b).

(b) In cases 1° and 2° the problem is reduced to problem 34 on p. 39 in NML 21. We consider the case 3°. Let S_3' be a circle with center O_3' that meets the required circle Σ' with center O' at A. In the triangle $O'AO_3'$ we know the sides $O_3'A$ and $O'A$ (the latter is equal to $\frac{r_1 \pm r_2}{2}$, where r_1 and r_2 are the radii of S_1' and S_2') and the angle $O'AO_3'$; we can therefore determine the distance $O_3'O'$ and then find Σ'. If S_3' is a line that intersects the circle Σ' with center O' in points A and B, then we know the angles and lateral sides of the isosceles triangle $O'AB$; this enables us to determine its altitude, the distance of O' from S_3', and then to construct Σ'.[1]

Second solution of problem (a). Let A_1, A_2, and A_3 be the points in which the required circle Σ touches the given circles S_1, S_2, and S_3 (Figure 171; we restrict ourselves to the case in which S_1, S_2, and S_3 are circles). Then the line A_1A_2 passes through the center of similarity O_1 of the circles S_1 and S_2, and the inversion with center at O_1—which takes S_1 to S_2—takes A_1 to A_2 (see problem 1 in Section 1 and its solution). In much the same way, the line A_2A_3 passes through the center of similarity O_2 of the circles S_2 and S_3, and the inversion with center at O_2—which takes S_2 to S_3—takes A_2 to A_3; finally, the line A_3A_1 passes through the center of similarity O_3 of the circles S_3 and S_1, and the inversion with center at O_1—which takes S_3 to S_1—takes A_3 to A_1.

Now we apply the sequence of inversions I_1, I_2, I_3. This sequence takes S_1 to itself and fixes A_1. We can find A_1 as in the solution of problem 20; the fact that in the solution of problem 20 each of the inversions $I_1, I_2, \ldots, I_n$ took S to itself is irrelevant. Since three is an odd number, we must introduce the reflection I in S_1 and seek on S_1 a point A_1 fixed by the sequence of inversions I_1, I_2, I_3, I. Once A_1 has been found, it is easy to construct Σ.

Each of the inversions I_1, I_2, I_3, which interchange two given circles, can be chosen in two ways (see the text in fine print on p. 10; if S_1 and S_2

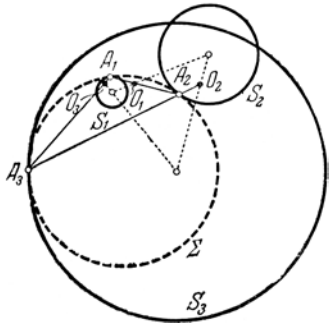


FIGURE 171

are congruent, then, instead of the inversion I_1, we can take the reflection in the perpendicular to the segment joining the centers of S_1 and S_2, erected at its midpoint). This implies that the problem can have up to eight solutions.

27. (a) This problem generalizes problems 25 and 26 and is solved in a manner similar to their solutions. We again consider the same three cases as those in the solution of the previous problem.

1°. S_1 and S_2 intersect one another. Then we can take them by inversion to two intersecting lines S_1' and S_2'; the inversion will take the required circle (or line) Σ to a circle (or line) Σ' which cuts the lines S_1' and S_2' at given angles α and β.

Let $\overline{\Sigma}'$ be an arbitrary circle. Let $\overline{S}'_1$ and $\overline{S}'_2$ be two lines forming the same angle as the circles S_1' and S_2', and let these two lines cut Σ' at angles α and β (Figure 172a). The angles, formed with the lines $\overline{S}'_1$ and $\overline{S}'_2$ by the tangents $\bar{t}_1$ and $\bar{t}_2$ to $\overline{\Sigma}'$, drawn from the point of intersection of $\overline{S}'_1$ and $\overline{S}'_2$, depend only on α and β and not on the radius of $\overline{\Sigma}'$; hence we can determine them by constructing arbitrarily $\overline{\Sigma}'$ and then the lines $\overline{S}'_1$ and $\overline{S}'_2$. Now we draw through the point of intersection of S_1' and S_2' lines t_1 and t_2 which form the same angles with S_1' and S_2' as $\bar{t}_1$ and $\bar{t}_2$ with $\overline{S}'_1$ and $\overline{S}'_2$; the lines t_1 and t_2 will be tangent to $\overline{\Sigma}'$. Thus our problem is reduced to the construction of a circle Σ', tangent to the known lines t_1 and t_2, and cutting at a given angle γ the circle (or line) S_3', which is the image of S_3 under our inversion, that is, to a special case of problem 26(b).

2°. S_1 and S_2 are tangent to one another. In this case, we can take S_1 and S_2 by inversion to two parallel lines S_1' and S_2'; here the required circle (or line) Σ will go over to a circle Σ' which cuts the parallel lines S_1' and S_2' at known angles α and β.

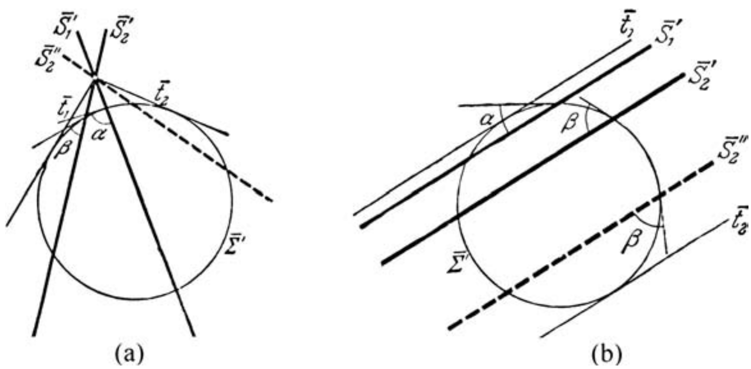


FIGURE 172

Let $\overline{\Sigma}'$ be an arbitrary circle and let $\overline{S}'_1$ and $\overline{S}'_2$ be two parallel lines that cut $\overline{\Sigma}'$ at angles α and β (Figure 172b). We draw tangents $\overline{t}_1$ and $\overline{t}_2$ parallel to $\overline{S}'_1$ and $\overline{S}'_2$. Clearly, the distances between the lines $\overline{t}_1$ and $\overline{t}_2$ and $\overline{S}'_1$ and $\overline{S}'_2$ depend only on the angles α and β and not on the circle $\overline{\Sigma}'$; we can determine them by drawing an arbitrary circle $\overline{\Sigma}'$ and then the lines $\overline{S}'_1$ and $\overline{S}'_2$. After constructing lines t_1 and t_2 such that the quartet of lines S'_1, S'_2, t_1, t_2 is similar to the quartet of lines $\overline{S}'_1$, $\overline{S}'_2$, $\overline{t}_1$, $\overline{t}_2$ we again arrive at the problem of constructing a circle Σ' tangent to known (parallel) lines t_1 and t_2 that cut the circle (or line) S'_3—the image of S_3 under our inversion—at a known angle γ.

3°. S_1 and S_2 are disjoint. In this case we take S_1 and S_2 by inversion to two concentric circles S'_1 and S'_2; here the required circle (or line) Σ will go over to a circle (or line) Σ' which cuts S'_1 and S'_2 at angles α and β.

We construct the circle Σ' which cuts S'_1 and S'_2 at angles α and β and intersects S'_1 in the given point M (see problem 24(b)[2]). The problem of constructing Σ' can have up to four solutions; the radius of Σ' and the distance between the centers of Σ' and S'_1 depend only on the circles S'_1 and S'_2 and on the angles α and β and not on the position of M. That is why the circles that cut the concentric circles S'_1 and S'_2 at angles α and β form, in general, four families, each of which consists of distinct circles whose centers are at the same distance from the common center of S'_1 and S'_2. We must find the circles of those families which cut the given circle (or line) S'_3 at the given angle γ. By making use of the fact that all circles of a given radius which cut the circle (or line) S'_3 at a given angle γ touch a circle $\overline{S}'_3$ concentric with S'_3 (or line $\overline{S}'_3$ parallel to S'_3) we can easily construct Σ', and then the required circle Σ.

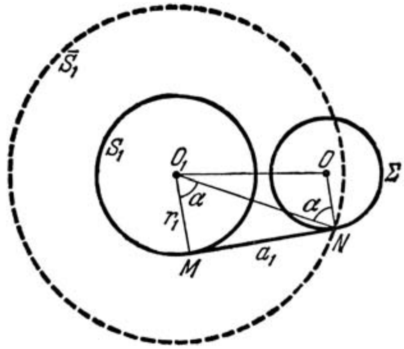


FIGURE 174

The problem has up to eight solutions; if S_1', S_2', and S_3' are lines, then the problem can turn out to be indeterminate.

(b) This problem can be reduced to the previous problem by the following device. Let the length of the common tangent MN of a circle Σ and a given circle S_1 with center O_1 and radius r_1 be equal to a given segment a_1; we assume for definiteness that MN is the common outer tangent of the circles S_1 and Σ (Figure 174). Then the point N is on a circle $\overline{S}_1$ concentric with S_1 with radius $\sqrt{r_1^2 + a_1^2}$. Since the radius ON of Σ is perpendicular to MN, the angle α between the circles $\overline{S}_1$ and Σ has the known value $\angle ONO_1 = \angle NO_1M$ $(= \arctan \frac{a_1}{r_1})$. In much the same way we can construct two more circles $\overline{S}_2$ and $\overline{S}_3$, concentric with S_2 and S_3 respectively, with which the required circle Σ forms known angles. Then all we need do is construct the circle Σ which forms given angles with the circles S_1, S_2, and S_3, that is, we have reduced problem (b) to problem (a).

28. We draw a circle with center at B and radius BA and make on it marks M, N, C such that $AM = MN = NC = BA$ (Figure 175). Clearly, AM,

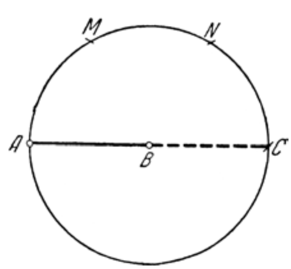


FIGURE 175

MN, and NC are sides of a hexagon inscribed in the circle; hence C and A are diametrically opposite points of the circle, that is, C, B, and A are collinear and $AB = BC$.

29. Let A be outside Σ (Figure 176a). We draw a circle with center A and radius AO (O is the center of Σ); let M and N be the points of intersection of this circle with the circle Σ. We draw two circles with centers M and N and radii MO and NO. The point A', common to these two circles, is the required point: in fact, the similarity of the isosceles triangles AOM and MOA' with common base angle implies that $\frac{AO}{OM} = \frac{OM}{OA'}$, whence $OA \cdot OA' = OM^2$, which is what we were to show. [We note that this construction is simpler than the construction in the text (p. 33), which uses both ruler and compass.]

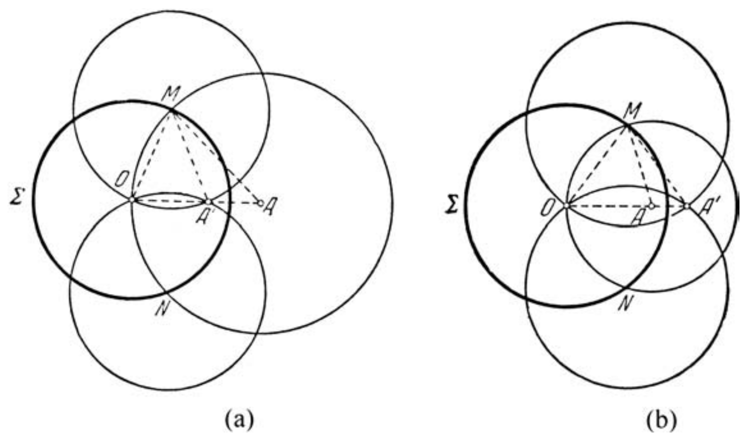


FIGURE 176

If A is inside Σ, then our construction can be carried out only if the circle with center A and radius AO intersects Σ in two points (that is, if the distance OA from A to the center of Σ is greater than half the radius of Σ; see Figure 176b). However, for any point A we can find an integer n such that $n \cdot OA$ is greater than half the radius of Σ. We find on the extension of OA beyond A a point B such that $OB = n \cdot OA$ (see the remark in the text following problem 28), and then we construct the point B' symmetric to B with respect to S (in Figure 177, $n = 2$). Now if A' is a point on the line OA such that $OA' = n \cdot OB'$, then $OA \cdot OA' = OB \cdot OB'$, that is, A' is the required point symmetric to A with respect to Σ.

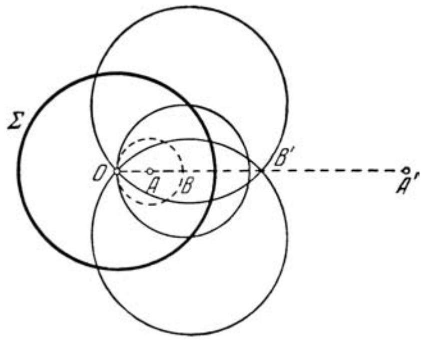


FIGURE 177

30. We draw any circle Σ with center at a point A of the given circle S which intersects S at points P and Q; further, let K be a point symmetric to A with respect to the line PQ (Figure 178; it is obvious how to construct K with compass alone). The similarity of the isosceles triangles APK and AOP (O is the required center of S) with common base angle implies that $\frac{AP}{AK} = \frac{AO}{AP}$, that is, $AK \cdot AO = AP^2$. Since O is symmetric to K with respect to the circle Σ, we can construct it (see the previous problem).

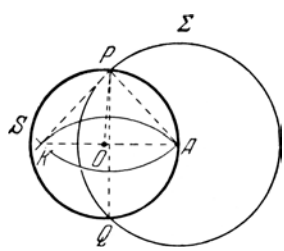


FIGURE 178

31. The solution of this problem is very similar to the solution of the previous problem. We draw the circle Σ with center at A and radius AB (Figure 179). The point B plays the role of the point P in the previous construction. But we do not know the second point B_1 of the intersection of Σ with the required circle S, which is why we are not able to construct the point K symmetric to A with respect to the common chord of S and Σ. To get around this difficulty, we note that the image of S under reflection in Σ is the line BB_1 (for S passes through the center of Σ); hence the point C', symmetric to C with respect to Σ (see problem 29) is on that line. Knowing

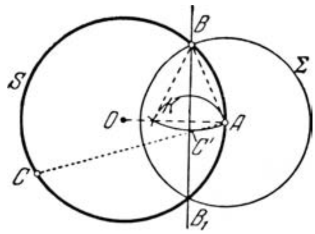


FIGURE 179

B and C' we can easily find K, symmetric to A with respect to the chord BC'. The subsequent construction of the center of S is the same as in the previous problem; having found its center we can construct S itself.

32. (a) *First solution.* We take three points A, B, and C on the line l (when it comes to finding the third point on the line it is useful to look at problem 28), obtain the points A', B', and C' symmetric to A, B, and C with respect to the circle Σ (see problem 29), and then construct the circle S passing through A', B', and C' (problem 31). Clearly, S is the required circle.

Second solution. Let O be the center of Σ (problem 30), K the point symmetric to O with respect to the line AB, and O_1 the point symmetric to K with respect to Σ (Figure 180). We claim that O_1 is the center of the required circle S; we can now construct S using its center O_1 and one of its points O. In fact, let K_1 be the point of intersection of lines OK and l, and let L be the second point of intersection of the required circle S with the line OK. The points L and K_1 are symmetric with respect to Σ (see the proof of property B_2 of inversion, p. 8), and since $OL = 2O\overline{O}_1$, where $\overline{O}_1$ is the center of S, $OK_1 = \frac{1}{2}OK$, it follows that $\overline{O}_1$ and K are also symmetric with respect to Σ, that is, $\overline{O}_1$ coincides with O_1.

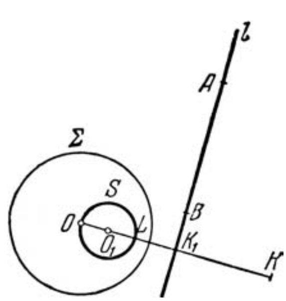


FIGURE 180

(b) *First solution.* Let A, B, and C be three points on the circle S and let A', B', and C' be symmetric to A, B, and C with respect to the circle Σ (problem 29). Clearly, the circle S' passing through A', B', and C' (problem 31) is the required circle.

Second solution. Let O and O_1 be the centers of circles Σ and S, R and r their radii, $d = OO_1$ the distance between the centers of Σ and S, K the point symmetric to O with respect to S (problem 29), and O_1' symmetric to K with respect to Σ (Figure 181). We claim that O_1' is the center of the required circle S'; we can construct S' if we know its center O_1' and one of its points A' (symmetric to some point A of S with respect to Σ).[3] In fact, $O_1O \cdot O_1K = r^2$, whence $O_1K = \frac{r^2}{d}$, $KO = d \pm \frac{r^2}{d} = \frac{d^2 \pm r^2}{d}$. Using also the equality $OK \cdot OO_1' = R^2$ we obtain

$$\frac{OO_1'}{OO_1} \cdot \frac{OK}{KO_1} = \frac{R^2}{r^2}, \quad \frac{OO_1'}{OO_1} = \frac{R^2}{r^2} \cdot \frac{KO_1}{OK} = \frac{R^2}{r^2} \cdot \frac{r^2}{d^2 \pm r^2};$$

and finally

$$\frac{OO_1'}{OO_1} = \frac{k}{k_1}$$

where

$$k = R^2, \quad k_1 = d^2 \pm r^2.$$

Thus O_1' is centrally similar to O_1 with center of similarity O and coefficient of similarity $\frac{k}{k_1}$, which implies our assertion (see the proof of property B_4 of inversion, p. 9).

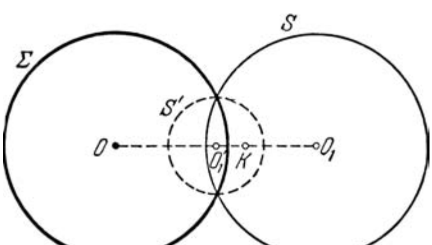


FIGURE 181

Notes to Section 2

[1] One can show that the circle Σ' with known radius $\frac{r_1 \pm r_2}{2}$, which intersects the given circle (or line) S_3', is tangent to a definite circle (or line) $\overline{S}_3'$ concentric with (or parallel to) S_3'.

[2] Here is yet another construction. Let N be the point of intersection of $\overline{\Sigma}'$ and S_2', O_1 and O the centers of S_1' (and S_2') and $\overline{\Sigma}'$ (Figure 173). We rotate the triangle ONO_1 about O through the angle NOM into position OMO_1'. We can construct the point O_1' because we know the distance $MO_1' = NO_1$ and the angle O_1MO_1' (for $\angle OMO_1 = \alpha$, $\angle OMO_1' = \angle ONO_1 = \beta$); then it is easy to find O (for $\angle O_1MO = \alpha$ and $O_1O = O_1'O$).

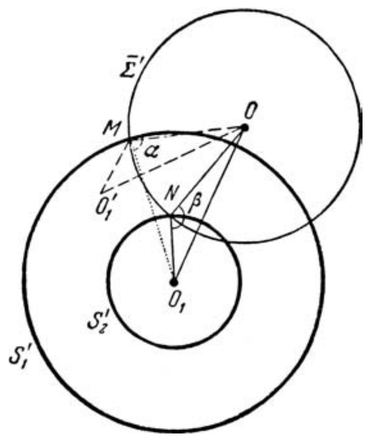


FIGURE 173

[3] Figure 181 represents the case when S and Σ intersect one another; in that (simpler) case we can take for A' a point of intersection of S and Σ.

Section 3

33. The assertion of the problem is a special case of the theorem: The three radical axes of three circles taken two at a time meet in a point or are parallel (see p. 53).

34. Since the common chord is the radical axis of two circles, the tangents from any of its points to the circles are equal.

35. If P is a point of the required locus, then $PA \cdot PB = PM^2$ (a property of a tangent and secant of the circle Σ). Hence the power $PA \cdot PB$ of P with respect to S is equal to the power PM^2 of this point with respect to the point M (see p. 50). It follows that P is on the radical axis r of the point M and the circle S.

36. (a) If S is a line parallel to AB, then the required circle Σ is tangent to S at the midpoint of the segment A_1B_1, where A_1 and B_1 are the projections of A and B on S (Figure 182a). If S is a line that intersects AB at M, then we know the power $MA \cdot MB$ of M with respect to the required circle Σ, and thus the length of the segment MT between M and the point of tangency of S and Σ; the problem can have two solutions (Figure 182b). Thus it remains to consider the case when S is a circle. We draw an arbitrary circle $\overline{S}$ passing through A and B and intersecting S at points M and N (Figure 182c). The radical axis of the required circle Σ and of $\overline{S}$ is the line AB; the radical axis of $\overline{S}$ and S is the line MN; and the radical axis of Σ and S is their common tangent t. Hence t is the tangent to S passing through the point Z of intersection of lines AB and MN, the radical center of $S, \overline{S}$, and Σ (and if $MN \parallel AB$, then, likewise, $t \parallel AB$). The problem can have two solutions, one solution, or no solution.

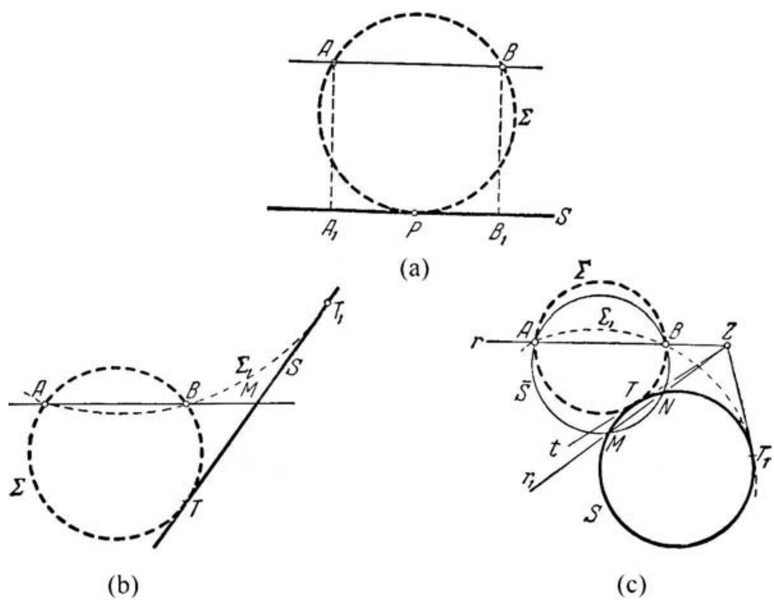


FIGURE **182**

(b) If S_1 and S_2 do not intersect one another, then all circles perpendicular to S_1 and S_2 pass through two points A and B (see p. 46); hence problem (b) reduces to problem (a). [In order to determine A and B it suffices to draw two circles perpendicular to S_1 and S_2; as the centers of

these circles we can take any two points of the radical axis of S_1 and S_2, and their radii are equal to the lengths of the tangents drawn from their centers to S_1 and S_2.] If S_1 and S_2 are tangent to one another, then all circles perpendicular to S_1 and S_2 pass through their point of tangency A and the centers of all these circles lie on the radical axis r of S_1 and S_2 (which passes through A); also, the center of the required circle Σ is equidistant from the center O of the circle S and from the point D on the line r such that AD is equal to the radius of S (or from the line S and from the line through A perpendicular to r). If S_1 and S_2 intersect one another and S is a line parallel to the line l of the centers of S_1 and S_2 (the radical axis of the pencil perpendicular to S_1 and S_2), then the required circle Σ is tangent to S at the point P of intersection of S and the radical axis r of S_1 and S_2

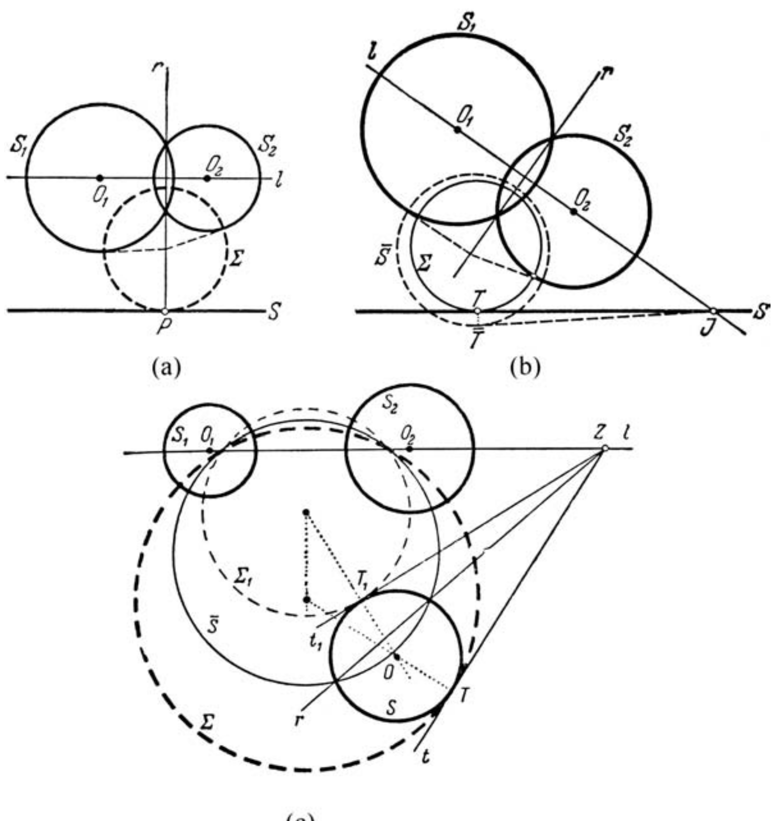


FIGURE 183

(the line of centers of the pencil perpendicular to S_1 and S_2; Figure 183a); its center is the point of intersection of r and the radical axis r_1 of the circle S_1 and the point P. If the line S intersects l at a point J, and $\overline{S}$ is a circle perpendicular to S_1 and S_2, then the tangent $\overline{T}$, drawn from J to $\overline{S}$, is equal to the tangent JT, drawn from J to the required circle Σ (for l is the radical axis of $\overline{S}$ and Σ) (Figure 183b; compare this with the solution of problem (a)); the problem has two solutions. If S is a circle and $\overline{S}$ is some circle perpendicular to S_1 and S_2 (Figure 183c), then the radical axis of $\overline{S}$ and the required circle Σ is the line of centers of S_1 and S_2; one can construct the radical axis r of $\overline{S}$ and S. Hence the radical axis t of the circles Σ and S—their common tangent—passes through the point Z of intersection of l and r, or is parallel to l if $l \parallel r$ (compare this with the solution of problem (a)).

The problem can have two solutions, one solution, or no solution; if S_1 and S_2 are tangent to one another, and S passes through their point of tangency and is perpendicular to them, then the problem is undetermined.

37. (a) *First solution.* The circles perpendicular to S_1 and S_2 form a pencil; we are required to find the circle of this pencil passing through M. If S_1 and S_2 do not intersect one another, then the circles of the pencil pass through two definite points A and B (see the solution of problem 36(b)); the required circle S passes through A, B, and M. If S_1 and S_2 are tangent at A, then the center of the required circle S lies on the intersection of the common tangent r of the circles S_1 and S_2 (their radical axis) and the perpendicular to the segment AM erected at its midpoint. Finally, if S_1 and S_2 intersect one another in points A and B (Figure 184), then we draw the circle S through A, B, and M— a circle of the pencil which includes S_1

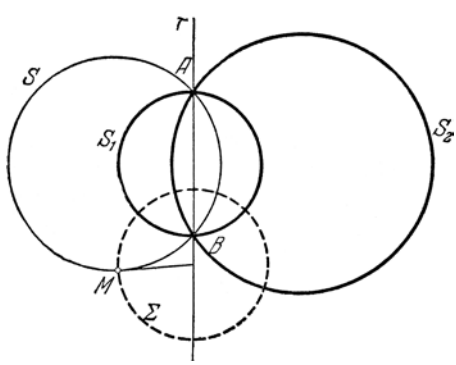


FIGURE 184

and S_2. The required circle Σ is perpendicular to the whole pencil; hence the center of Σ is the point of intersection of the radical axis of S_1 and S_2 and the tangent to S at M.

Second solution. The center of the required circle Σ coincides with the point of intersection of the radical axis r of the circles S_1 and S_2 and the radical axis r_1 of the circle S_1 and the point M (a "circle of zero radius"; r_1 passes through the midpoints of the tangents drawn from M to S_1).

(b) The center of the required circle Σ is the point of intersection of the radical axis r of the circle S_1 and the point M, and the line s, symmetric to the radical axis r_1 of the circle S_2 and the point M with respect to the midpoint of the segment O_2M (O_2 is the center of S_2; see the text in fine print on pp. 54–55).

(c) The center of the required circle Σ coincides with the point of intersection of the line s, symmetric to the radical axis r of the circles S_1 and S_2 with respect to the midpoint of the segment O_1O_2 (O_1 and O_2 are the centers of S_1 and S_2), and the line s_1, symmetric to the radical axis r_1 of the circle S_1 and the point M with respect to the midpoint of the segment O_1M.

38. (a) The center of the required circle Σ coincides with the radical center of the circles S_1, S_2, and S_3 (see p. 53); its radius is equal to the length of the tangent drawn from O to S_1. The problem has one solution if O is outside S_1, S_2, and S_3 and none otherwise.

(b) The center of the required circle Σ coincides with the point of intersection of the lines s_1 and s_2, symmetric to the radical axes r_3 and r_2 of the circles S_1 and S_2, S_1 and S_3, with respect to the midpoints of the respective segments O_1O_2 and O_1O_3 (O_1, O_2, and O_3 are the centers of the circles S_1, S_2, and S_3; see the text in fine print on pp. 54–55); Σ passes through the points of intersection of S_1 with the perpendicular to OO_1 erected at O_1. The problem always has a unique solution.

(c) The center of the required circle Σ coincides with the radical center of the circles S_1, S_2, and S_3 (see the text in fine print on pp. 55–56); Σ intersects S_1 in points A and B such that AB passes through O and $AB \perp OO_1$ (O_1 is the center of S_1). The problem has a unique solution if O is in the interior of S_1, S_2, and S_3 and none otherwise.

39. (a) Let M be a point of the required locus, Q the projection of M on the line of centers O_1O_2 of the given circles S_1 and S_2 with radii r_1 and r_2 (Figure 185a). By arguments similar to those on pp. 50–53 we obtain the

equality

$$2O_1O_2 \cdot TQ = r_1^2 - r_2^2 + a,$$

where T is the midpoint of O_1O_2. It follows that the required locus is a line perpendicular to the line of centers ($TQ = \frac{r_1^2 - r_2^2 + a}{2O_1O_2}$ is known), and thus parallel to the radical axis.

(b) The formulas $2O_1O_2 \cdot TP = r_1^2 - r_2^2$ (see p. 52) and $2O_1O_2 \cdot TQ = r_1^2 - r_2^2 + a$ (see the solution of problem (a)) imply that

$$2O_1O_2 \cdot QP = a.$$

In words: *The difference a of the powers of a point M with respect to two circles S_1 and S_2 is equal to twice the product of the distance O_1O_2 between the centers of S_1 and S_2 by the distance $QP = MP'$ from M to the radical axis r of S_1 and S_2* (Figure 185a).

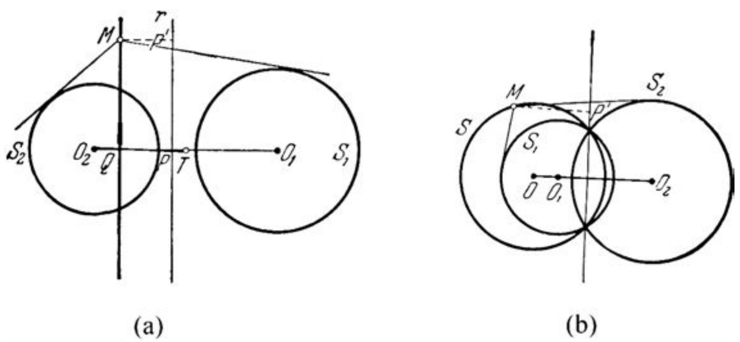


FIGURE 185

Now let S be a circle in the same pencil as S_1 and S_2, M a point on S, and O_1, O_2, and O the centers of the circles S_1, S_2, and S (Figure 185b). Then *the power of M with respect to S is zero*. Therefore, by the theorem just stated,

$$2OO_1 \cdot MP' = a_1, \qquad 2OO_2 \cdot MP' = a_2,$$

where a_1 and a_2 are the powers of M with respect to S_1 and S_2. Hence

$$\frac{a_1}{a_2} = \frac{OO_1}{OO_2}.$$

In words: *the ratio of the powers of a point M with respect to circles S_1 and S_2 is equal to the distance from the center O of a circle in the pencil of*

circles containing S_1 *and* S_2 *to the centers of* S_1 *and* S_2 (here we assume that M is not a point on the radical axis r of the circles S_1 and S_2; we recall that through every point not on r there passes a unique circle of the pencil containing S_1 and S_2; see p. 46).

It follows that the required locus is a circle if $k \neq 1$ and a line if $k = 1$. Knowing k one can easily construct the required locus.

40. (a) Let l pass through the center of similarity O of the circles S_1 and S_2 (see Figure 57a in the text). Then the central similarity with center O takes S_1 to S_2 and l to itself; hence l forms equal angles α with S_1 and S_2. This implies that the tangents to S_1 and S_2 at points A_1 and A_2, which correspond to one another under the central similarity, are parallel; the tangents to S_1 and S_2 at B_1 and B_2 are likewise parallel. Further, let M be the point of intersection of the tangents to S_1 and S_2 at A_1 and B_2; then $\angle MA_1B_2 = \angle MB_2A_1 = \alpha$, whence $MA_1 = MB_2$, that is, M is on the radical axis of S_1 and S_2. In much the same way one proves that the tangents to S_1 and S_2 at B_1 and A_2 meet in a point N on the radical axis.

(b) Let the tangents to S_1 and S_2 at A_1 and A_2 meet in a point M (see Figure 57b in the text). Let α_1 and α_2 be the angles formed by l with S_1 and S_2. Then

$$\frac{MA_1}{MA_2} = \frac{\sin \angle MA_2A_1}{\sin \angle MA_1A_2} = \frac{\sin \alpha_2}{\sin \alpha_1},$$

which implies that M is on the circle Σ, the locus of points such that the ratio of their powers with respect to S_1 and S_2 is $\frac{\sin^2 \alpha_2}{\sin^2 \alpha_1}$ (see problem 39). In much the same way we prove that the remaining three points of intersection of the tangents to S_1 at A_1, B_1 with the tangents to S_2 at A_2, B_2 also lie on Σ.

Remark. The result in problem 39(b) implies that the circles Σ, S_1, and S_2 in problem 40(b) belong to the same pencil.

41. Let $A_1A_2A_3 \ldots A_n$ be an n-gon inscribed in S and circumscribed about s. We are to show that if we move A_1 on S and consider the continuously varying n-gon $A_1'A_2'A_3' \ldots A_n'$ all of whose vertices lie on S and all of whose sides except its last side $A_n'A_1'$ are tangent to s, then that side is also tangent to s.

Let $A_1A_2A_3 \ldots A_n$ and $A_1'A_2'A_3' \ldots A_n'$ be two close positions of the varying n-gon (Figure 186). We consider the quadrilateral $A_1A_1'A_2A_2'$. Let R_1 and R_2 be the points in which the line M_1M_1', joining the points M_1 and M_1' of tangency of A_1A_2 and $A_1'A_2'$ with s, intersects A_1A_1' and A_2A_2'.

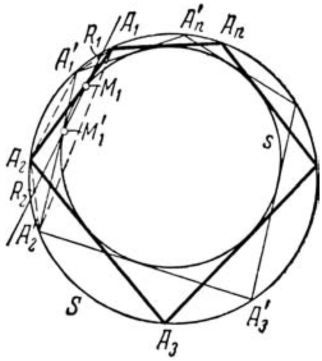


FIGURE 186

Figure 186 tells us that

$$\angle A_1R_1R_2 = \angle R_1M_1'A_1' + \angle R_1A_1'A_2',$$
$$\angle A_2'R_2R_1 = \angle R_2M_1A_2 + \angle R_2A_2A_1.$$

But $\angle R_1A_1'A_2' = \angle R_2A_2A_1$ (both are subtended by the arc A_1A_2' of the circle S) and $\angle R_1M_1'A_1' = \angle R_2M_1A_2$ (both are subtended by the arc M_1M_1' of the circle s). Hence

$$\angle A_1R_1R_2 = \angle A_2'R_2R_1,$$

which implies that there exists a circle s' tangent to A_1A_1' and A_2A_2' at R_1 and R_2. Also, the circle passing through the points of intersection of the tangents to s and s' at M_1, M_1', and R_1, R_2, that is, the circle S passing through the points A_1, A_1', A_2, A_2', belongs to the same pencil as the circles s and s' (see the remark at the end of the solution of problem 40(b)); in other words, *s' belongs to the same pencil as the circles S and s.*

In much the same way we prove that there is a circle tangent to A_2A_2' and A_3A_3' at the points of intersection of A_2A_2' and A_3A_3' with the line joining the points at which A_2A_3 and $A_2'A_3'$ are tangent to s; this circle—denote it by s''—also belongs to the pencil which includes S and s. It is very important to note that *the circle s'' coincides with the circle s'.* In fact, there are, in general, two circles of the pencil in question—the pencil containing S and s—tangent to the line A_2A_2' (see the solution of problem 36(b); this follows from the fact that every pencil of circles can be taken by inversion to a set of concentric circles, or parallel lines, or lines passing through a point). Let us assume that s' and s'' are two different circles, and let us change the n-gon $A_1'A_2'A_3' \dots A_n'$ continuously so that

it tends to the n-gon $A_1A_2A_3\ldots A_n$. As we do this, the circles s' and s'' also change continuously; they will therefore tend to different circles of the pencil containing S and s, circles tangent to the line tangent to S at A_2 (the line A_2A_2' tends to this very tangent). But this contradicts the fact that s' and s'' tend to one and the same circle S (for A_1A_1' and A_2A_2' tend to the tangents to S at A_1 and A_2).

We have shown that the circle s' is tangent to the lines A_1A_1', A_2A_2', and A_3A_3'. In much the same way we show that it is also tangent to the lines A_4A_4', $A_5A_5', \ldots, A_nA_n'$. Thus we see that there exists a circle s' tangent to A_1A_1' and A_nA_n'. Now we consider the quadrilateral $A_1A_1'A_nA_n'$; we show—proceeding as we did earlier—that there exists a circle $\overline{s}$, belonging to the same pencil as s and s', tangent to A_1A_n and $A_1'A_n'$ (at the points of intersection of these lines with the line joining the points R_1 and R_n at which A_1A_1' and A_nA_n' are tangent to s'). But A_1A_n is tangent to the circle s of this pencil. This implies that during the process of change of the n-gon the side A_1A_n remains tangent to the same circle s. [A_1A_n is also tangent to a circle s_1 of the pencil containing s, S, and s'; but, as is easy to see, the points of intersection of R_1R_n with A_1A_n lie inside the segments A_1A_n and $A_1'A_n'$, which implies that $\overline{s}$ coincides with s and not with s_1.]

Remark. In much the same way we can prove the following more general theorem: *If an n-gon $A_1A_2A_3\ldots A_n$ changes continuously but is at all times inscribed in a circle S so that its sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ remain tangent to circles $s_1, s_2, \ldots, s_{n-1}$ in a single pencil which includes S, then its last side A_nA_1 remains tangent to a circle s_n in the same pencil.* Also, every diagonal of the n-gon $A_1A_2A_3\ldots A_n$ remains tangent to a circle in the same pencil; in particular, *if, while changing continuously, the n-gon remains inscribed in a circle S and circumscribed about a circle s, then each of its diagonals remains tangent to a circle in a single pencil which includes S and s.*

Section 4

42. (a) We apply an inversion with center at the point M and power 1. Then the points $A_1, A_2, A_3, \ldots, A_n$ go over to collinear points $A_1', A_2', A_3', \ldots, A_n'$. Let a be the length of a side of the regular n-gon (Figure 187). Formula ($*$) (p. 61) implies that

$$A_1'A_2' = \frac{1}{d_1d_2}a,$$

$$A_2'A_3' = \frac{1}{d_2d_3}a,$$

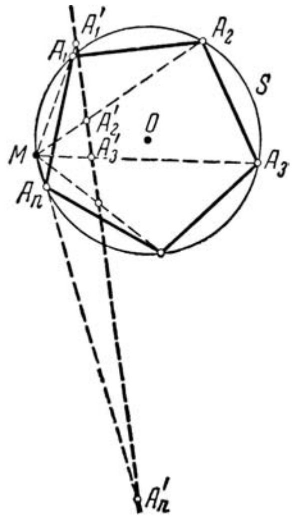


FIGURE 187

$$A_3'A_4' = \frac{1}{d_3d_4}a, \ \ldots, \ A_{n-1}'A_n' = \frac{1}{d_{n-1}d_n}a,$$
$$A_1'A_n' = \frac{1}{d_1d_n}a.$$

Substituting all these expressions in the obvious relation

$$A_1'A_n' = A_1'A_2' + A_2'A_3' + A_3'A_4' + \cdots + A_{n-1}'A_n'$$

and dividing both sides of the resulting equality by a we obtain the required result.

(b) We denote by a a side of the regular n-gon and by b a chord which subtends two sides of the n-gon. Let $A_1', A_2', A_3', \ldots, A_n'$ be the images of the vertices of the n-gon under the inversion with center M and power 1. Then we have (see the solution of problem (a)):

$$A_1'A_2' = \frac{1}{d_1d_2}a,$$
$$A_2'A_3' = \frac{1}{d_2d_3}a, \ldots, A_1'A_n' = \frac{1}{d_1d_n}a;$$
$$A_1'A_3' = \frac{1}{d_1d_3}b,$$

$$A_2'A_4' = \frac{1}{d_2 d_4} b, \ldots, A_{n-2}'A_n' = \frac{1}{d_{n-2} d_n} b,$$
$$A_{n-1}'A_1' = \frac{1}{d_1 d_{n-1}} b,$$
$$A_n'A_2' = \frac{1}{d_2 d_n} b.$$

Further,

$$A_1'A_2' + A_2'A_3' = A_1'A_3', \quad \text{that is,} \quad \frac{1}{d_1 d_2} a + \frac{1}{d_2 d_3} a = \frac{1}{d_1 d_3} b,$$
$$\text{or} \quad ad_1 + ad_3 = bd_2,$$
$$A_2'A_3' + A_3'A_4' = A_2'A_4', \quad \text{that is,} \quad \frac{1}{d_2 d_3} a + \frac{1}{d_3 d_4} a = \frac{1}{d_2 d_4} b,$$
$$\text{or} \quad ad_2 + ad_4 = bd_3,$$
$$A_3'A_4' + A_4'A_5' = A_3'A_5', \quad \text{that is,} \quad \frac{1}{d_3 d_4} a + \frac{1}{d_4 d_5} a = \frac{1}{d_3 d_5} b,$$
$$\text{or} \quad ad_3 + ad_5 = bd_4,$$
$$A_4'A_5' + A_5'A_6' = A_4'A_6', \quad \text{that is,} \quad \frac{1}{d_4 d_5} a + \frac{1}{d_5 d_6} a = \frac{1}{d_4 d_6} b,$$
$$\text{or} \quad ad_4 + ad_6 = bd_5,$$

....

$$A_{n-2}'A_{n-1}' + A_{n-1}'A_n' = A_{n-2}'A_n', \quad \text{that is,}$$
$$\frac{1}{d_{n-2} d_{n-1}} a + \frac{1}{d_{n-1} d_n} a = \frac{1}{d_{n-2} d_n} b, \quad \text{or} \quad ad_{n-2} + ad_n = bd_{n-1},$$
$$A_{n-1}'A_n' + A_{n-1}'A_1' = A_n'A_1', \quad \text{that is,}$$
$$\frac{1}{d_{n-1} d_n} a + \frac{1}{d_{n-1} d_1} b = \frac{1}{d_n d_1} a, \quad \text{or} \quad ad_1 + bd_n = ad_{n-1},$$
$$A_n'A_2' + A_2'A_1' = A_1'A_n', \quad \text{that is,}$$
$$\frac{1}{d_2 d_n} b + \frac{1}{d_1 d_2} a = \frac{1}{d_1 d_n} a, \quad \text{or} \quad bd_1 + ad_n = ad_2.$$

Adding all “final” equalities we obtain the equality

$$(2a + b)(d_1 + d_3 + d_5 + \cdots + d_n) = (2a + b)(d_2 + d_4 + \cdots + d_{n-1}),$$

which implies the required result.

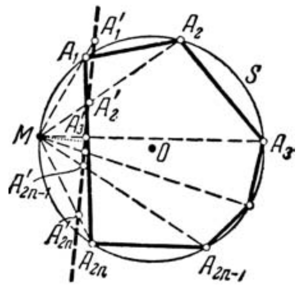


FIGURE 188

43. (a) We apply an inversion with center M. Then the vertices $A_1, A_2, A_3, \ldots, A_{2n}$ of the inscribed $2n$-gon go over to $2n$ collinear points $A_1', A_2', A_3', \ldots, A_{2n}'$ (Figure 188). The similarity of the triangles MA_1A_2 and $MA_2'A_1'$ (see p. 61) implies that

$$\frac{p_1}{p'} = \frac{MA_1}{MA_2'} = \frac{MA_2}{MA_1'}, \quad \text{or that} \quad \frac{p_1^2}{p'^2} = \frac{MA_1 \cdot MA_2}{MA_1' \cdot MA_2'};$$

here p' is the distance from M to the line $A_1', A_2', A_3', \ldots, A_{2n}'$, that is, the altitude of the triangle $MA_1'A_2'$. Similarly,

$$\frac{p_2^2}{p'^2} = \frac{MA_2 \cdot MA_3}{MA_2' \cdot MA_3'},$$

$$\frac{p_3^2}{p'^2} = \frac{MA_3 \cdot MA_4}{MA_3' \cdot MA_4'}, \ldots, \frac{p_{2n-1}^2}{p'^2} = \frac{MA_{2n-1} \cdot MA_{2n}}{MA_{2n-1}' \cdot MA_{2n}'},$$

$$\frac{p_{2n}^2}{p'^2} = \frac{MA_{2n} \cdot MA_1}{MA_{2n}' \cdot MA_1'},$$

whence

$$\left(\frac{p_1p_3p_5\cdots p_{2n-1}}{p'^n}\right)^2 = \frac{MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 \cdots MA_{2n-1} \cdot MA_{2n}}{MA_1' \cdot MA_2' \cdot MA_3' \cdot MA_4' \cdots MA_{2n-1}' \cdot MA_{2n}'}$$

and

$$\left(\frac{p_2p_4p_6\cdots p_{2n}}{p'^n}\right)^2 = \frac{MA_2 \cdot MA_3 \cdot MA_4 \cdot MA_5 \cdots MA_{2n} \cdot MA_1}{MA_2' \cdot MA_3' \cdot MA_4' \cdot MA_5' \cdots MA_{2n}' \cdot MA_1'},$$

that is,

$$p_1p_3p_5\cdots p_{2n-1} = p_2p_4p_6\cdots p_{2n},$$

which is what we had to prove.

(b) This theorem can be viewed as a limiting case of the theorem in problem (a); if the vertex A_2 of the inscribed $2n$-gon tends to the vertex A_1, the vertex A_3—to the vertex A_4, the vertex A_5—to the vertex A_6, and so on, then the sides A_1A_2, A_3A_4, $A_5A_6, \ldots$ of the inscribed $2n$-gon tend to the tangents to the circle at the vertices $A_1, A_2, A_3, \ldots$ of the inscribed n-gon $A_1A_3A_5 \ldots A_{2n-1}$.

44. We apply an inversion with center at M. The vertices $A_1, A_2, A_3, \ldots,$ A_n of the n-gon will go over to n collinear points $A'_1, A'_2, A'_3, \ldots, A'_n$ (Figure 188); moreover,

$$A'_1A'_n = A'_1A'_2 + A'_2A'_3 + A'_3A'_4 + \cdots + A'_{n-1}A'_n. \qquad (*)$$

We denote by p' the length of the perpendicular dropped from M to the line $A'_1A'_n$, the common altitude of the triangles $A'_1MA'_2, A'_2MA'_3, \ldots,$ $A'_{n-1}MA'_n$, and $A'_1MA'_n$. The similarity of the triangles A_1MA_2 and $A'_1MA'_2$ implies that

$$\frac{A_1A_2}{A'_1A'_2} = \frac{p_1}{p'}$$

—the ratio of corresponding sides of the similar triangles is equal to the ratio of the altitudes dropped to these sides. It follows that

$$A'_1A'_2 = \frac{A_1A_2}{p_1}p' = \frac{a_1}{p_1}p',$$

similarly,

$$A'_2A'_3 = \frac{a_2}{p_2}p', \; A'_3A'_4 = \frac{a_3}{p_3}p', \ldots, A'_{n-1}A'_n = \frac{a_{n-1}}{p_{n-1}}p', \; A'_1A'_n = \frac{a_0}{p_0}p'.$$

Substituting all these expressions in $(*)$ and dividing the resulting equality by p' we obtain the required result.

45. We apply an inversion with center A_0 and power 1. The circles Σ and S_0 will go over to parallel lines Σ' and S'_0, and Figure 60 in the text to Figure 189. We denote the radii of the transformed circles $S'_1, S'_2, \ldots, S'_n$

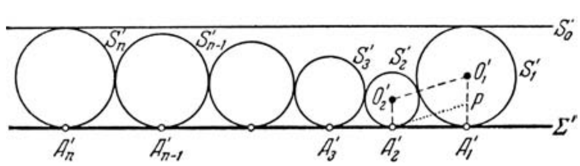


FIGURE 189

by $r_1', r_2', \ldots, r_n'$; clearly, $r_1' = r_n'$. From Figure 189 we obtain:

$$A_1'A_2'^2 = O_1'O_2'^2 - A_1'P^2 = (r_1' + r_2')^2 - (r_1' - r_2')^2 = 4r_1'r_2',$$

and, similarly,

$$A_2'A_3'^2 = 4r_2'r_3', \quad A_3'A_4'^2 = 4r_3'r_4', \ \ldots, A_{n-1}'A_n'^2 = 4r_{n-1}'r_n'.$$

(a) Let n be even. We have:

$$r_2' = \frac{A_1'A_2'^2}{4r_1'}, \qquad r_3' = \frac{A_2'A_3'^2}{4r_2'} = \frac{A_2'A_3'^2}{A_1'A_2'^2}r_1',$$

$$r_4' = \frac{A_3'A_4'^2}{4r_3'} = \frac{A_1'A_2'^2 \cdot A_3'A_4'^2}{A_2'A_3'^2}\frac{1}{4r_1'}, \ldots,$$

$$r_n' = \frac{A_1'A_2'^2 \cdot A_3'A_4'^2 \cdots A_{n-1}'A_n'^2}{A_2'A_3'^2 \cdot A_4'A_5'^2 \cdots A_{n-2}'A_{n-1}'^2}\frac{1}{4r_1'}.$$

The condition $r_n' = r_1'$ enables us to determine the diameter of the circle S_1'—the distance between the lines Σ' and S_0':

$$2r_1' = \frac{A_1'A_2' \cdot A_3'A_4' \cdots A_{n-1}'A_n'}{A_2'A_3' \cdot A_4'A_5' \cdots A_{n-2}'A_{n-1}'}$$

or, since

$$\begin{aligned} A_1'A_2' &= \frac{A_1A_2}{A_0A_1 \cdot A_0A_2}, \\ A_2'A_3' &= \frac{A_2A_3}{A_0A_2 \cdot A_0A_3}, \ldots, \\ A_{n-1}'A_n' &= \frac{A_{n-1}A_n}{A_0A_{n-1} \cdot A_0A_n} \end{aligned} \tag{$**$}$$

(see the formula $(*)$ on p. 61),

$$2r_1' = \frac{A_1A_2 \cdot A_3A_4 \cdots A_{n-1}A_n}{A_0A_1 \cdot A_2A_3 \cdot A_4A_5 \cdots A_{n-2}A_{n-1} \cdot A_nA_0}.$$

Let d be the distance from the center of inversion A_0 to the line Σ'; then the distance from A_0 to S_0' will be $d \pm 2r_1'$. In that case, $2R = \frac{1}{d}$, $2r_0 = \frac{1}{d \pm 2r_1'}$ (see the proof of property B_3 of inversion), and, finally, we obtain

$$d = \frac{1}{2R}, \qquad r_0 = \frac{1}{\frac{1}{R} \pm 2\frac{A_1A_2 \cdot A_3A_4 \cdots A_{n-1}A_n}{A_0A_1 \cdot A_2A_3 \cdot A_4A_5 \cdots A_{n-2}A_{n-1} \cdot A_nA_0}}$$

(the plus sign corresponds to the case of inner tangency of $S_0, S_1, \ldots, S_n$ and S and the minus sign to that of outer tangency).

(b) Just as in the solution of problem (a) we obtain:

$$r'_n = \frac{A'_2A'^2_3 \cdot A'_4A'^2_5 \cdots A'_{n-1}A'^2_n}{A'_1A'^2_2 \cdot A'_3A'^2_4 \cdots A'_{n-2}A'^2_{n-1}} r'_1,$$

which implies the following relation between the mutual distances of the points $A'_1, A'_2, \ldots, A'_n$:

$$A'_1A'_2 \cdot A'_3A'_4 \cdots A'_{n-2}A'_{n-1} = A'_2A'_3 \cdot A'_4A'_5 \cdots A'_{n-1}A'_n.$$

Using the formulas ($**$) in the solution of problem (a), we obtain the required relation between the mutual distances of the points $A_0, A_1, A_2, \ldots, A_n$:

$$A_1A_2 \cdot A_3A_4 \cdots A_{n-2}A_{n-1} \cdot A_nA_0 = A_0A_1 \cdot A_2A_3 \cdot A_4A_5 \cdots A_{n-1}A_n.$$

46. Clearly, the required locus is characterized by the fact that the cross ratio $\frac{AM_1}{BM_1} : \frac{AM_2}{BM_2}$, where M_1 and M_2 are any two points of the locus, has the value 1. By property D, this implies that if an inversion takes the points A, B to points A', B', then it takes the required locus to a locus of points such that the ratio of their distances to A' and B' is constant (although this ratio need not be equal to the ratio of the distances from the points of the initial locus to A and B).

Now suppose that the center O of the inversion is a point of the required locus. Then the ratio of the distances from a point M', which is the image under the inversion of a point M of the locus, to the points A' and B' is equal to 1:

$$M'A' : M'B' = MA\frac{k}{OM \cdot OA} : MB\frac{k}{OM \cdot OB} = \frac{MA}{MB} : \frac{OA}{OB} = 1$$

(see the formula ($*$) on p. 61; $\frac{MA}{MB} = \frac{OA}{OB}$ by the choice of the point O). It follows that the image of the required locus consists of points equidistant from A' and B', and thus is a line. Hence the required locus is a line or a circle.

Clearly, the required locus is a line only if the ratio stated in the condition of the problem has the value 1.

47. The ratio of the sum of the products of the opposite sides of a quadrilateral $ABCD$ to the product of its diagonals,

$$\frac{AB \cdot CD + AD \cdot BC}{AC \cdot BD},$$

can be written as

$$\frac{AB \cdot CD}{AC \cdot BD} + \frac{AD \cdot BC}{AC \cdot BD} = \frac{AB}{DB} : \frac{AC}{DC} + \frac{DA}{CA} : \frac{DB}{CB},$$

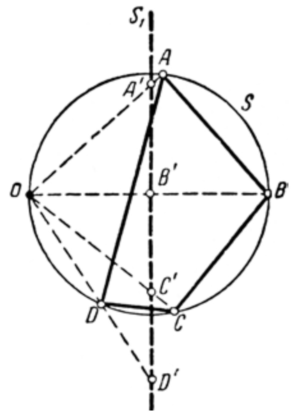


FIGURE 190

and is therefore the sum of the cross ratios of the quadruples A, D, B, C and D, C, A, B (note the order!); this being so, it is unchanged by inversion.

Suppose that one can circumscribe a circle S about the quadrilateral $ABCD$. We apply an inversion with center at any point O of S; then the points A, B, C, D go over to collinear points A', B', C', D'.

Assume for definiteness that the points A', B', C', D' appear on the line in the indicated order (Figure 190; this will be the case if the point O lies on the arc AD of S); let a, b, and c denote the lengths of the segments $A'B'$, $B'C'$, and $C'D'$. Then

$$A'D' = a + b + c, \quad A'C' = a + b, \quad B'D' = b + c,$$

whence

$$\frac{A'B' \cdot C'D' + A'D' \cdot B'C'}{A'C' \cdot B'D'} = \frac{ac + (a+b+c)b}{(a+b)(b+c)}$$
$$= \frac{ac + ab + b^2 + bc}{ab + b^2 + ac + bc} = 1,$$

which proves the assertion of the problem.

48. Let P, Q, R, T be any four points on the circle S, and let P', Q', R', T' be their images under the sequence of inversions $I_1, I_2, \ldots, I_n$ (see the solution of problem 20). Since inversions preserve cross ratios, it follows that

$$\frac{PR}{QR} : \frac{PT}{QT} = \frac{P'R'}{Q'R'} : \frac{P'T'}{Q'T'}.$$

But, clearly,

$$PR = 2r \sin\frac{\overset{\smile}{PR}}{2}, \quad QR = 2r\sin\frac{\overset{\smile}{QR}}{2}, \text{ and so on,}$$

where r is the radius of the circle S; hence the equality of the ratios just stated can be rewritten as follows:

$$\frac{\sin\frac{\overset{\smile}{PR}}{2}}{\sin\frac{\overset{\smile}{QR}}{2}} : \frac{\sin\frac{\overset{\smile}{PT}}{2}}{\sin\frac{\overset{\smile}{QT}}{2}} = \frac{\sin\frac{\overset{\smile}{P'R'}}{2}}{\sin\frac{\overset{\smile}{Q'R'}}{2}} : \frac{\sin\frac{\overset{\smile}{P'T'}}{2}}{\sin\frac{\overset{\smile}{Q'T'}}{2}}.$$

Finally, let O be any fixed point of S. Then $\overset{\smile}{PR} = \overset{\smile}{PO} - \overset{\smile}{RO}$,[1] and so on; hence

$$\begin{aligned}\sin\frac{\overset{\smile}{PR}}{2} &= \sin\frac{\overset{\smile}{PO}-\overset{\smile}{RO}}{2} = \sin\frac{\overset{\smile}{PO}}{2}\cos\frac{\overset{\smile}{RO}}{2} - \sin\frac{\overset{\smile}{RO}}{2}\cos\frac{\overset{\smile}{PO}}{2}\\ &= \cos\frac{\overset{\smile}{PO}}{2}\cos\frac{\overset{\smile}{RO}}{2}\left(\tan\frac{\overset{\smile}{PO}}{2} - \tan\frac{\overset{\smile}{RO}}{2}\right),\end{aligned}$$

and so on. From this we obtain:

$$\begin{aligned}&\frac{\tan\frac{\overset{\smile}{PO}}{2} - \tan\frac{\overset{\smile}{RO}}{2}}{\tan\frac{\overset{\smile}{QO}}{2} - \tan\frac{\overset{\smile}{RO}}{2}} : \frac{\tan\frac{\overset{\smile}{PO}}{2} - \tan\frac{\overset{\smile}{TO}}{2}}{\tan\frac{\overset{\smile}{QO}}{2} - \tan\frac{\overset{\smile}{TO}}{2}}\\ &\qquad = \frac{\tan\frac{\overset{\smile}{P'O}}{2} - \tan\frac{\overset{\smile}{R'O}}{2}}{\tan\frac{\overset{\smile}{Q'O}}{2} - \tan\frac{\overset{\smile}{R'O}}{2}} : \frac{\tan\frac{\overset{\smile}{P'O}}{2} - \tan\frac{\overset{\smile}{T'O}}{2}}{\tan\frac{\overset{\smile}{Q'O}}{2} - \tan\frac{\overset{\smile}{T'O}}{2}}.\end{aligned}$$

Now assume that T is the point A_1, which we denote by X (an unknown); then T' also coincides with A_1 (the sequence of n inversions takes A_1 to itself). Hence

$$\begin{aligned}&\frac{\tan\frac{\overset{\smile}{PO}}{2} - \tan\frac{\overset{\smile}{RO}}{2}}{\tan\frac{\overset{\smile}{QO}}{2} - \tan\frac{\overset{\smile}{RO}}{2}} : \frac{\tan\frac{\overset{\smile}{PO}}{2} - \tan\frac{\overset{\smile}{XO}}{2}}{\tan\frac{\overset{\smile}{QO}}{2} - \tan\frac{\overset{\smile}{XO}}{2}}\\ &\qquad = \frac{\tan\frac{\overset{\smile}{P'O}}{2} - \tan\frac{\overset{\smile}{R'O}}{2}}{\tan\frac{\overset{\smile}{Q'O}}{2} - \tan\frac{\overset{\smile}{R'O}}{2}} : \frac{\tan\frac{\overset{\smile}{P'O}}{2} - \tan\frac{\overset{\smile}{XO}}{2}}{\tan\frac{\overset{\smile}{Q'O}}{2} - \tan\frac{\overset{\smile}{XO}}{2}}.\end{aligned} \tag{$*$}$$

We choose arbitrarily points P, Q, R; now it won't be difficult to find P', Q', R'. The equality $(*)$ can be viewed as a quadratic equation in the unknown $\tan\frac{\overset{\smile}{XO}}{2}$. By solving this equation we obtain $\tan\frac{\overset{\smile}{XO}}{2}$ and can therefore construct the central angle corresponding to the arc $\overset{\smile}{XO}$ and find the point

$X = A_1$; after that it is a simple matter to obtain the remaining vertices of the n-gon. The construction is easily done using ruler and compass.

Depending on the number of roots of the equation $(*)$ the problem has two solutions, one solution, or no solution.

49. This problem is very close to problem 46. First of all, it is obvious that if the ratio of the lengths of the tangents drawn from a point M_1 to two circles S_1 and S_2 is equal to the ratio of the lengths of the tangents drawn from another point M_2 to the same circles, then the cross ratio

$$\frac{t_{S_1M_1}}{t_{S_2M_1}} : \frac{t_{S_1M_2}}{t_{S_2M_2}}$$

of the four circles S_1, S_2, M_1, M_2 (here we are viewing the points M_1 and M_2 as "circles of zero radius"; for example, $t_{S_1M_1}$ denotes the length of the segment of the tangent drawn from the point M_1 to the circle S_1) is equal to 1. It follows that if an inversion with center outside S_1 and S_2 takes these circles to circles S_1' and S_2', then the required locus goes over to a locus of points such that the ratio of the lengths of the tangents drawn from these points to the circles S_1' and S_2' has constant magnitude.

Now let O be any point of the required locus; since one can draw from it tangents to S_1 and and S_2, it must be outside these circles. *An inversion with center at O takes the required locus to a locus of points such that the tangents from these points to the image circles S_1' and S_2' are equal.* In fact, if a point M_1 of the required locus goes over to a point M_1', then, by formula $(**)$ (see p. 68), we have:

$$\frac{t_{S_1'M_1'}}{t_{S_2'M_1'}} = \left(t_{S_1M_1}\frac{k}{\sqrt{OM_1^2\cdot k_1}}\right) : \left(t_{S_2M_1}\frac{k}{\sqrt{OM_1^2\cdot k_2}}\right)$$

$$= \frac{t_{S_1M_1}}{t_{S_2M_1}} : \frac{\sqrt{k_1}}{\sqrt{k_2}} = \frac{t_{S_1M_1}}{t_{S_2M_1}} : \frac{\sqrt{t^2_{S_1O}}}{\sqrt{t^2_{S_2O}}} = \frac{t_{S_1M_1}}{t_{S_2M_1}} : \frac{t_{S_1O}}{t_{S_2O}} = 1$$

(the powers k_1 and k_2 of O with respect to S_1 and S_2 are $t^2_{S_1O}$ and $t^2_{S_2O}$; $\frac{t_{S_1O}}{t_{S_2O}} = \frac{t_{S_1M_1}}{t_{S_2M_1}}$ by the choice of O). Thus we see that under an inversion with center O the required locus goes over to a locus of points M' such that $t_{M'S_1'} = t_{M'S_2'}$, that is, to a segment of the radical axis of S_1' and S_2' exterior to both of these circles (see Section 3, p. 49).

It follows that the required locus is a circle or part of a circle (an arc of the circle passing through the points of intersection of S_1 and S_2 located in

the exterior of S_1 and S_2); if the value of the ratio in the statement of the problem is 1, then the required locus is a line or part of a line (part of the common chord of the circles in the exterior of both of them).

50. This problem is very close to problem 47. It is easy to see that if S_1, S_2, S_3, and S_4 are four circles, then the expression

$$\frac{t_{12}t_{34} + t_{14}t_{23}}{t_{13}t_{24}}$$

can be rewritten as

$$\frac{t_{21}}{t_{31}} : \frac{t_{24}}{t_{34}} + \frac{t_{41}}{t_{31}} : \frac{t_{42}}{t_{32}},$$

which shows that it is unchanged by an inversion whose center is either outside the circles S_1, S_2, S_3, and S_4 or inside all of them.

If the circles S_1, S_2, S_3, and S_4 are tangent to a line Σ at points A, B, C, and D (Figure 191), then the relation involved in the present problem takes the form

$$\frac{AB \cdot CD + AD \cdot BC}{AC \cdot BD} = 1;$$

it is easy to verify that this equality actually holds (see the solution of problem 47).

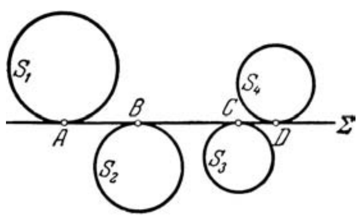


FIGURE 191

Now suppose that the four circles S_1, S_2, S_3, and S_4 are tangent to a circle Σ. If, say, S_1 contains Σ in its interior and S_2 does not, then either S_1 and S_2 have same-name tangency with Σ but S_1 and S_2 have no common outer tangent (Figure 192a), or S_1 and S_2 have different-name tangency with Σ but S_1 and S_2 have no common inner tangent (Figure 192b). Hence the condition of the problem is meaningful only if Σ is either inside the circles S_1, S_2, S_3, and S_4 or outside all of them. Now let O be a point of Σ. An inversion with center at O takes the circles S_1, S_2, S_3, and S_4 to circles S_1', S_2', S_3', and S_4' tangent to a single line Σ', and for these, as we

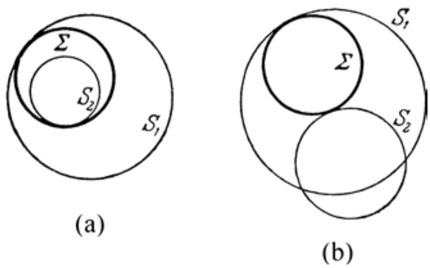


FIGURE 192

saw earlier, the condition of the problem holds. But then this condition must also hold for the four initial circles.

51. Let S_1 and S_2 be two circles which intersect one another at a point A, let O_1 and O_2 be their centers, r_1 and r_2 their radii ($r_1 \geq r_2$), and let $MN = t_{12}$ be the segment of their common tangent (Figure 193). Let d denote the distance O_1O_2. Then

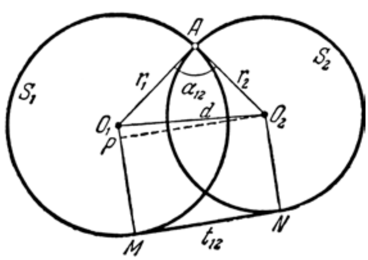


FIGURE 193

$$t_{12}^2 = d^2 - (r_1 - r_2)^2 = d^2 - r_1^2 - r_2^2 + 2r_1r_2.$$

Hence

$$\frac{t_{12}^2}{r_1r_2} = \frac{2r_1r_2 - (r_1^2 + r_2^2 - d^2)}{r_1r_2} = 2 - \frac{r_1^2 + r_2^2 - d^2}{r_1r_2}.$$

A look at the triangle O_1O_2A tells us that

$$\frac{r_1^2 + r_2^2 - d^2}{r_1r_2} = 2\frac{O_1A^2 + O_2A^2 - O_1O_2^2}{O_1A \cdot O_2A} = 2\cos\alpha_{12},$$

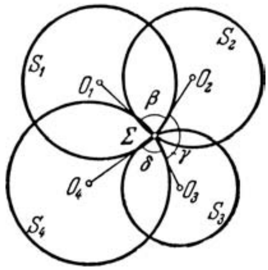


FIGURE 194

where $\alpha_{12} = \angle O_1AO_2$ is the angle between the radii O_1A and O_2A of the circles S_1 and S_2. Hence [2]

$$\frac{t_{12}^2}{r_1r_2} = 2(1 - \cos\alpha_{12}) = 4\sin^2\frac{\alpha_{12}}{2}. \qquad (*)$$

Now we assume that the four circles S_1, S_2, S_3, and S_4 meet in a point Σ. Let the centers of these circles be S_1, S_2, S_3, and S_4 (Figure 194) and let the pairwise angles between the lines $O_1\Sigma$, $O_2\Sigma$, $O_3\Sigma$, and $O_4\Sigma$ (the angles between the circles) be $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$.

$$\alpha_{12} = \beta, \quad \alpha_{13} = \beta + \gamma, \quad \alpha_{14} = \beta + \gamma + \delta,$$
$$\alpha_{23} = \gamma, \quad \alpha_{24} = \gamma + \delta, \quad \alpha_{34} = \delta,$$

hence

$$\begin{aligned}
\sin\frac{\alpha_{12}}{2}\sin\frac{\alpha_{34}}{2} &+ \sin\frac{\alpha_{14}}{2}\sin\frac{\alpha_{23}}{2} - \sin\frac{\alpha_{13}}{2}\sin\frac{\alpha_{24}}{2} \\
&= \sin\frac{\beta}{2}\sin\frac{\delta}{2} + \sin\frac{\beta+\gamma+\delta}{2}\sin\frac{\gamma}{2} - \sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\delta}{2} \\
&= \sin\frac{\beta}{2}\sin\frac{\delta}{2} + \left(\sin\frac{\beta+\gamma}{2}\cos\frac{\delta}{2} + \cos\frac{\beta+\gamma}{2}\sin\frac{\delta}{2}\right)\sin\frac{\gamma}{2} \\
&\quad - \sin\frac{\beta+\gamma}{2}\left(\sin\frac{\gamma}{2}\cos\frac{\delta}{2} + \cos\frac{\gamma}{2}\sin\frac{\delta}{2}\right) \\
&= \sin\frac{\beta}{2}\sin\frac{\delta}{2} + \cos\frac{\beta+\gamma}{2}\sin\frac{\delta}{2}\sin\frac{\gamma}{2} - \sin\frac{\beta+\gamma}{2}\cos\frac{\gamma}{2}\sin\frac{\delta}{2} \\
&= \sin\frac{\beta}{2}\sin\frac{\delta}{2} - \left(\sin\frac{\beta+\gamma}{2}\cos\frac{\gamma}{2} - \cos\frac{\beta+\gamma}{2}\sin\frac{\gamma}{2}\right)\sin\frac{\delta}{2} \\
&= \sin\frac{\beta}{2}\sin\frac{\delta}{2} - \sin\frac{\beta+\gamma-\gamma}{2}\sin\frac{\delta}{2} = 0.
\end{aligned}$$

If we substitute in the relation just obtained for the sines of half angles between the circles expressions implied by $(*)$, then we obtain the relation

$$\frac{1}{2}\frac{t_{12}}{\sqrt{r_1r_2}}\frac{1}{2}\frac{t_{34}}{\sqrt{r_3r_4}}+\frac{1}{2}\frac{t_{14}}{\sqrt{r_1r_4}}\frac{1}{2}\frac{t_{23}}{\sqrt{r_2r_3}}-\frac{1}{2}\frac{t_{13}}{\sqrt{r_1r_3}}\frac{1}{2}\frac{t_{24}}{\sqrt{r_2r_4}}=0,$$

or, by dividing by $\frac{1}{4}\frac{1}{\sqrt{r_1r_2r_3r_4}}$ and transferring one term to the right side, the required relation

$$t_{12}t_{34}+t_{14}t_{23}=t_{13}t_{24}.$$

52. The relation in the statement of this problem was proved in the previous problem for the case when the circles S_1, S_2, S_3, and S_4 intersect in a single point. For the case when the circles S_1, S_2, S_3, and S_4 are tangent to a circle or line Σ, this relation follows directly from the proposition in problem 50. To obtain it, we need only substitute in the relation in the statement of problem 50 the expressions for t_{12}, and so on, obtained from the relation $(*)$ in the solution of problem 51 ($t_{12} = 2\sin\frac{\alpha_{12}}{2}\sqrt{r_1r_2}$, and so on).

53. (a) Since inversion does not take the center of a circle to the center of its image, the property of the bisectors of angles formed by intersecting lines appearing in the statement of the problem must be reformulated, so that the new statement involves only concepts preserved by inversion. It is easy to see how to do this: instead of speaking of the locus of the centers of circles tangent to two intersecting lines we must say that the circles in question must form right angles with one or the other of the angle bisectors. By applying an inversion to the new statement we obtain the following theorem: *Circles tangent to two intersecting circles S_1 and S_2 intersect perpendicularly one of the two circles Σ_1 and Σ_2 which pass through the points of intersection of S_1 and S_2 and halve the angle between these circles* (Figure 195). These two auxiliary circles are called bisectral circles of the two initial circles.

We note that no such theorem holds if the circles S_1 and S_2 do not intersect one another. In that case, we can take S_1 and S_2 by inversion to two concentric circles S_1' and S_2' (see Theorem 2 in Section 1, p. 22). Then the circles tangent to S_1' and S_2' form two families, one of which consists of circles perpendicular to a definite circle Σ', concentric with S_1' and S_2', and the other—of circles no three of which are perpendicular to a single circle. It follows that the circles tangent to S_1 and S_2 form two families, one of which consists of circles perpendicular to a definite circle Σ (perpendicular to the pencil determined by the circles S_1 and S_2), and the other—of circles no three of which are perpendicular to a single circle.

Finally, by taking two tangent circles to two parallel lines, it is easy to see that the circles tangent to two tangent circles form two families, one of which consists

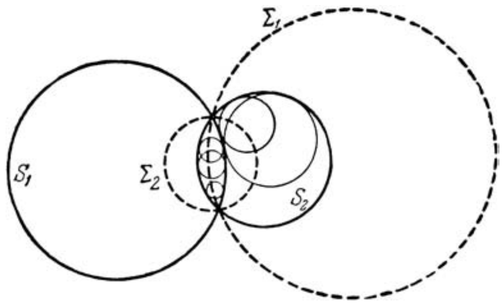


FIGURE 195

of circles passing through the point T of tangency of S_1 and S_2 and the other—of circles intersecting perpendicularly a certain circle Σ passing through T and tangent at T to S_1 and S_2.

(b) *Every circle that intersects two given intersecting circles S_1 and S_2 at equal angles is perpendicular to one of the two bisectral circles of S_1 and S_2* (see the solution of problem 53(a)); conversely, *every circle perpendicular to a bisectral circle of S_1 and S_2 intersects S_1 and S_2 at equal angles.*

(c) The result of problem (b) implies that our problem is equivalent to the following problem: Construct a circle which crosses perpendicularly three bisectral circles of the pairs $S_0, S_1,\ S_0, S_2,\ S_0, S_3$ of given circles S_0, S_1, S_2, S_3 (which can obviously be constructed), that is, to a special case of problem 38(a) in Section 3. This special case of problem 27(a) can have only one solution. Since each of the bisectral circles involved in the construction can be selected in one of two ways, the problem can have up to eight solutions.

54. Theorem: If three circles meet in a point, then the sum of the angles of the curvilinear triangle formed in their intersection (Figure 196) is equal to 180°.

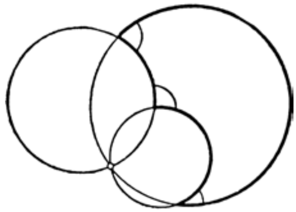


FIGURE 196

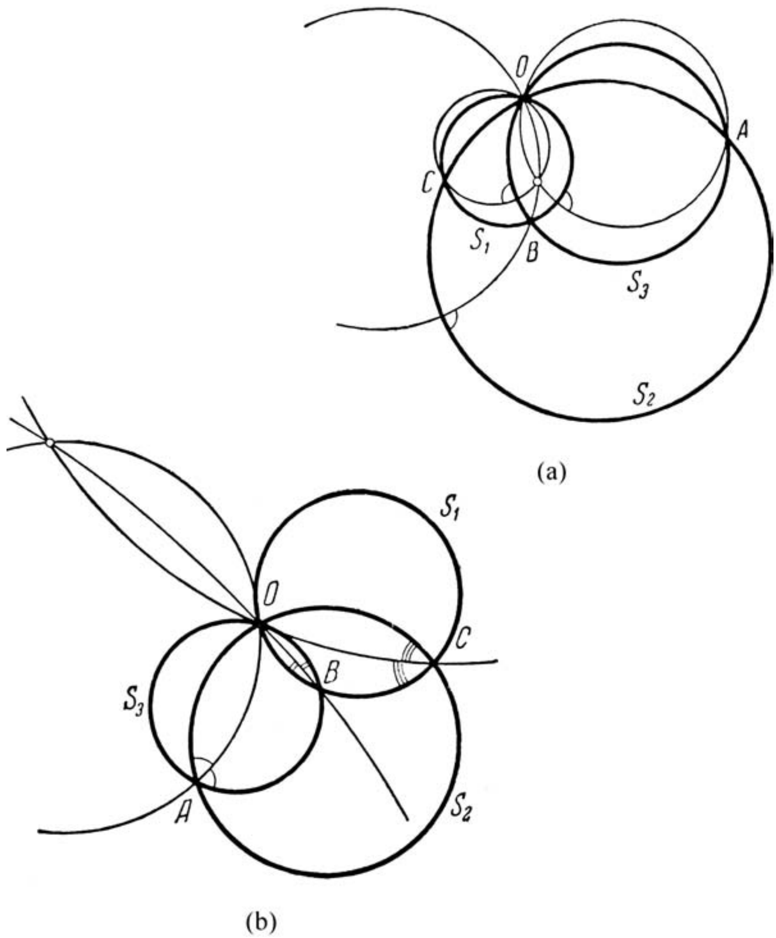


FIGURE 197

55. Let S_1, S_2, and S_3 be three circles passing though a point O; let A, B, and C be the second points of intersection of the pairs S_3 and S_2, S_1 and S_3, and S_2 and S_1. Then

(a) three circles that pass through the pairs of points O and A, O and B, and O and C perpendicularly to, respectively, S_1, S_2, and S_3 meet in a single point (Figure 197(a));

(b) three circles that pass through the pairs of points O and A, O and B, and O and C and bisect the angles between, respectively, S_3, S_2, S_1, S_3, and S_2, S_1 meet in a single point (Figure 197b).

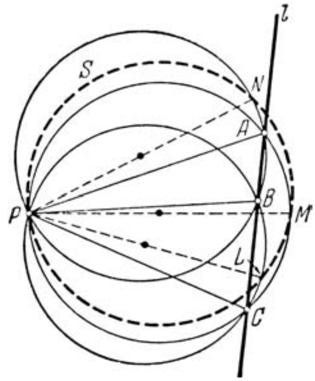


FIGURE 198

56. Let A, B, and C be three collinear points, and P a point not on that line. We circumscribe circles about the triangles PAB, PAC, and PBC. Let PN, PM, and PL be the diameters of these circles passing through P. Then P, N, M, and L are on a single circle S (Figure 198). Conversely, if PN, PM, and PL are three chords of a circle S, then the second points of intersection of the circles for which the segments PN, PM, and PL are diameters are collinear (see problem 62(b) in NML 21, p. 77).

57. An inversion whose center O does not lie on a given circle S takes S to some circle S'. Let Z' be the image of the center Z of the circle S, and let M' be the image of some point M on S (Figure 199). According to the relation $(*)$ on p. 61 we have:

$$M'Z' = MZ\frac{k}{OZ \cdot OM} = MZ\frac{OM'}{OZ}$$

(for $\frac{k}{OM} = OM'$). Whence

$$\frac{M'Z'}{M'O} = \frac{MZ}{OZ} = \frac{r}{OZ},$$

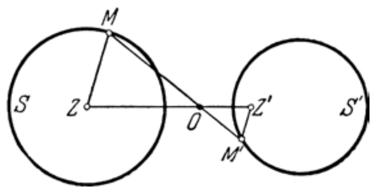


FIGURE 199

where r is the radius of S. Thus we see that the circle S' can be defined as the locus of points M' such that the ratio $\frac{M'Z'}{M'O}$ is equal to $\frac{r}{OZ}$, that is, it is independent of the choice of the point M' on the circle.

It follows that *a circle can be defined not only as the locus of points equidistant from a given point but also as the locus of points such that the ratio of their distances from two given points is constant* (cf. problem 46).

58. We apply an inversion with center at O. Let the images of the vertices A, B, C of the triangle ABC be A', B', C' (Figure 200a).

From (∗) on p. 61 we have:

$$AB = A'B'\frac{k}{OA' \cdot OB'}, \quad BC = B'C'\frac{k}{OB' \cdot OC'},$$

$$AC = A'C'\frac{k}{OA' \cdot OC'}.$$

Substituting these expressions in the inequality

$$AB + BC > AC$$

we obtain the relation

$$A'B'\frac{k}{OA' \cdot OB'} + B'C'\frac{k}{OB' \cdot OC'} > A'C'\frac{k}{OA' \cdot OC'},$$

or

$$A'B' \cdot OC' + B'C' \cdot OA' > A'C' \cdot OB'.$$

Since the points A, B, C are not collinear, the points O, A', B', C' are not concyclic; in other words, *if it is not possible to circumscribe a circle about*

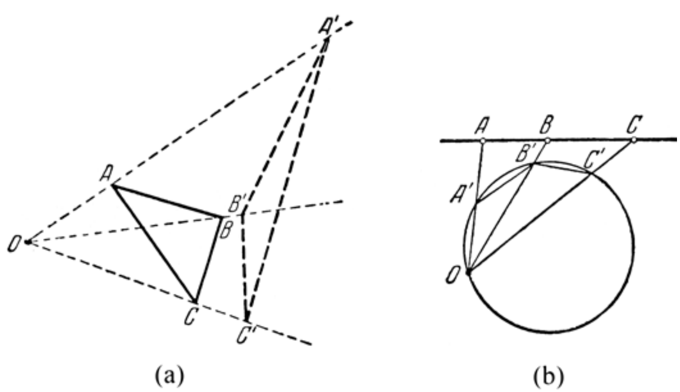


FIGURE 200

a quadrilateral, then the sum of the products of its opposite sides is greater than the product of its diagonals.

On the other hand, if the points A, B, C are collinear, that is, the points O, A', B', C' are concyclic (Figure 200b), then

$$AB + BC = AC,$$

whence

$$A'B'\frac{k}{OA' \cdot OB'} + B'C'\frac{k}{OB' \cdot OC'} = A'C'\frac{k}{OA' \cdot OC'},$$

or

$$A'B' \cdot OC' + B'C' \cdot OA' = A'C' \cdot OB'.$$

This implies Ptolemy's theorem as well as its converse.

59. (a) We apply an inversion with center at O. Let A', B', C' be the images of the vertices A, B, C of the right triangle ABC with right angle at B (Figure 201).

From the similarity of the triangles OAB and $OB'A'$, and of the triangles OBC and $OC'B'$, we have:

$$\angle OBA = \angle OA'B', \quad \angle OBC = \angle OC'B',$$

so that

$$\angle OA'B' + \angle OC'B' = 90°.$$

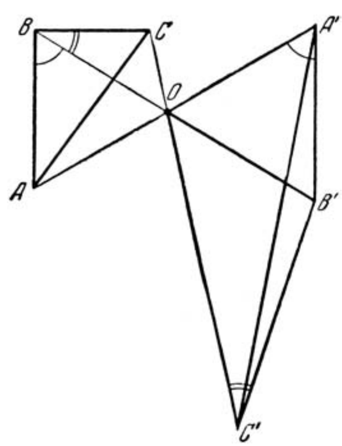


FIGURE 201

Further, by $(*)$ on p. 61,

$$AB = A'B'\frac{k}{OA' \cdot OB'},$$

$$BC = B'C'\frac{k}{OB' \cdot OC'}, \qquad AC = A'C'\frac{k}{OA' \cdot OC'}.$$

Using the theorem of Pythagoras $AB^2 + BC^2 = AC^2$ we obtain the relation

$$A'B'^2\frac{k^2}{OA'^2 \cdot OB'^2} + B'C'^2\frac{k^2}{OB'^2 \cdot OC'^2} = A'C'^2\frac{k^2}{OA'^2 \cdot OC'^2},$$

or

$$A'B'^2 \cdot OC'^2 + B'C'^2 \cdot OA'^2 = A'C'^2 \cdot OB'^2.$$

We have obtained the following theorem: *If the sum of the opposite angles of a convex quadrilateral is* 90°, *then the sum of the squares of the products of the opposite sides is equal to the product of its diagonals.*

It is interesting to juxtapose the result just obtained and Ptolemy's theorem, which can be stated as follows: *If the sum of the opposite angles of a convex quadrilateral is* 180°, *then the sum of the products of its opposite sides is equal to the product of its diagonals.*

(b) In much the same way as in the solution of the previous problem, using the law of cosines, we obtain the following generalization of both Ptolemy's theorem and the theorem in problem (a): *If the sum of the opposite angles in a quadrilateral is* φ, *then the sum of the products of the squares of the opposite sides minus twice the product of the sides by* $\cos\varphi$ *equals the product of the squares of the diagonals.* We leave the proof to the reader.

(c) As before, let A', B', C' be the images under inversion of the vertices of the triangle A, B, C (Figure 202). By the relation $(*)$ on p. 61,

$$AB = A'B'\frac{k}{OA' \cdot OB'},$$

$$BC = B'C'\frac{k}{OB' \cdot OC'}, \qquad AC = A'C'\frac{k}{OA' \cdot OC'}.$$

Now we obtain a connection between the angles of the triangle ABC and the angles of the quadrilateral $OA'B'C'$. First we have

$$\angle OBA = \angle OA'B', \quad \angle OBC = \angle OC'B',$$

hence

$$\angle ABC = \angle OA'B' + \angle OC'B',$$

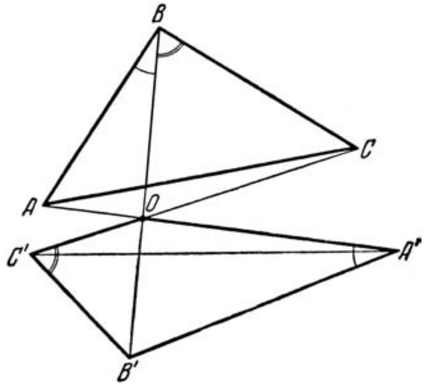


FIGURE 202

a fact used when solving problems (a) and (b). Further,

$$\angle OAB = \angle OB'A', \quad \angle OAC = \angle OC'A',$$

hence

$$\angle BAC = \angle OAB - \angle OAC = \angle OB'A' - \angle OC'A',$$

that is, the angle BAC of the triangle is equal to the difference of the angles formed by the diagonals OB' and $C'A'$ of the quadrilateral $OA'B'C'$ with its sides $B'A'$ and $C'O$. Similarly,

$$\angle BCA = \angle OB'C' - \angle OA'C',$$

that is, the angle BCA equals the difference of the angles formed by the diagonals OB' and $C'A'$ of the quadrilateral $OA'B'C'$ with its sides $B'C'$ and $A'O$ respectively. Substituting these expressions in the law of sines

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} = \frac{BC}{\sin A}$$

we obtain, after simplifying, the relation

$$\frac{A'C' \cdot OB'}{\sin\varphi} = \frac{A'B' \cdot OC'}{\sin\psi} = \frac{B'C' \cdot OA'}{\sin\chi}, \qquad (*)$$

where φ is the sum of the opposite angles $OA'B'$ and $OC'B'$ in the quadrilateral $OA'B'C'$, ψ is the difference of the two angles formed by the diagonals and the sides of the quadrilateral resting on the side OC', and χ is the difference of the two angles formed by the diagonals and the sides of the quadrilateral resting on the side OA'. The relation $(*)$ is the result of applying an inversion to the law of sines.

Notes to Section 4

[1] Here the rule for assigning signs to angles is analogous to the rule for reading angles in a trigonometric disk.

[2] From this it is clear that, for intersecting circles, the invariance of the expression $\frac{t_{12}}{\sqrt{r_1 r_2}}$ under inversion is a direct consequence of property C of inversion.

Section 5

60. We assume that the lines l_1 and l_2 and the circle S are directed. We apply a dilatation which takes the circle S to a point S'. This dilatation takes the lines l_1 and l_2 to lines l_1' and l_2' and the required circle Σ (which will also have to be viewed as directed), tangent to l_1, l_2, and S, to a circle Σ', tangent to the lines l_1' and l_2' and passing through the point S' (Figure 203). Thus the construction of the circle Σ, tangent to two given lines and a given circle, reduces to the construction of the circle Σ', tangent to two given lines and passing through a given point, that is, to problem 13(a) in NML 21, p. 18.

Problem 13(a) has, in general, two solutions. After assigning a direction to the circle S in an arbitrary way we can assign directions to the lines l_1 and l_2 in four ways. Hence the problem has, in general, eight solutions.

Clearly, this solution is simpler than the solutions of problems 13(c) and 22 in NML 21 (pp. 18 and 21) and of problem 21(b) in Section 2 in this book.

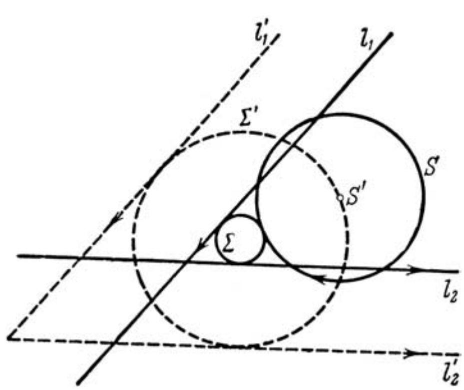


FIGURE 203

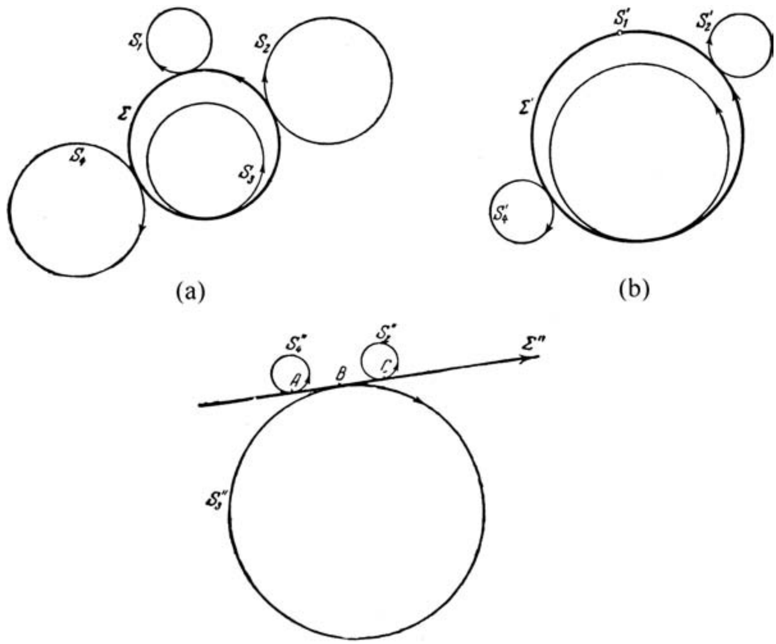
61. We consider the circles S_1, S_2, and S_3 directed. We apply a dilatation which takes one of these circles to a point (say, the circle S_1 to a point S_1'). Here the circles S_2 and S_3 will go over to circles S_2' and S_3', and the required (directed) circle Σ, tangent to S_1, S_2, and S_3, to a circle Σ' passing through the point S_1' and tangent to the circles S_2' and S_3'. Thus the problem of Apollonius is reduced to finding a circle Σ' tangent to the circles S_2' and S_3' and passing through the point S_1', that is, to problem 21(b) in Section 2 in this book. After solving this problem we will readily obtain the required circle.

Problem 21(b) reduces to finding a common tangent to two circles, and hence, in the case of directed circles, has, in general, two solutions. After assigning a direction to the circle S_1 in an arbitrary way we can assign directions to the circles l_2 and l_3 in four ways. Hence the problem of Apollonius has, in general, eight solutions.

It is easy to see that this solution of the problem of Apollonius is simpler than either one of the solutions of problem 26(a).

62. Clearly, this problem is a generalization of Ptolemy's theorem (see problem 47 in Section 4 in this book). In Section 3 we gave two proofs of Ptolemy's theorem using inversions. One of these proofs reduced the Ptolemy relation to the analogous relation $AB \cdot CD + AD \cdot BC = AC \cdot BD$, involving the distances between four collinear points taken two at a time; it enabled us to prove that the Ptolemy relation is necessary for four points to be concyclic (see problem 47). The second proof reduced the Ptolemy relation to the relation $AB + BC = AC$ connecting three collinear points taken two at a time; it enabled us to prove that the Ptolemy relation is necessary and sufficient for four points to be concyclic (see the solution of problem 58). The solution of problem 50 in Section 4 is analogous to the first proof of Ptolemy's theorem. Using a dilatation, one can also give a proof of the theorem in problem 50 analogous to the second proof of Ptolemy's theorem; here it turns out to be possible to prove the sufficiency of the condition in the theorem in problem 50 for four circles S_1, S_2, S_3, and S_4 to be tangent to a single circle (or line) Σ (that is, to prove the theorem in problem (b)). This is the content of the present problem.

We have four circles S_1, S_2, S_3, and S_4 (Figure 204a) which we suppose to be directed (this is equivalent to being told which of the magnitudes t_{12}, and so on, denote segments of common outer tangents and which denote segments of common inner tangents). We apply a dilatation which takes one of these circles to a point (say, the circle S_1 to a point S_1'), while the remaining three circles S_2, S_3, and S_4 go over to new circles S_2', S_3', and S_4'



(a) (b) (c)

FIGURE **204**

(Figure 204b). Then we apply an inversion with center at S_1'; this inversion takes the circles S_2', S_3', and S_4' to new circles S_2'', S_3'', and S_4'' (Figure 204c).

Further, we will prove separately the necessity and sufficiency of the hypothesis of the problem for the four circles S_1, S_2, S_3, and S_4 to be tangent to a single circle. The proof of necessity provides a solution of problem (a), and the proof of sufficiency provides a solution of problem (b).

Proof of necessity. If the circles S_1, S_2, S_3, and S_4 are tangent to a single circle or line Σ, or pass through a single point Σ, then the circles S_2', S_3', and S_4' are tangent to a single circle (or line) Σ' passing through S_1',[1] and the circles S_2'', S_3'', and S_4'' are tangent to a single line Σ''. Assume for definiteness that the point B of tangency of S_3'' and Σ'' lies between the points C and A of tangency of the circles S_2'' and S_3'' with Σ'' (Figure 204c); then

$$AB + BC = AC, \text{ or } t_{23}'' + t_{34}'' = t_{24}'',$$

where t''_{23} is the tangent distance between the circles S''_2 and S''_3 and the magnitudes t''_{34} and t''_{24} have analogous meanings.

Further, in view of the relation (∗∗) on p. 68,[2]

$$t''_{23} = t'_{23}\frac{k}{\sqrt{k_2k_3}}, \quad t''_{34} = t'_{34}\frac{k}{\sqrt{k_3k_4}}, \quad t''_{24} = t'_{24}\frac{k}{\sqrt{k_2k_4}},$$

where t''_{23}, t''_{34}, and t''_{24} are the tangent distances of the circles S'_2, S'_3, and S'_4 taken two at a time, k_2, k_3, and k_4 are the squares of the lengths of the tangents drawn from the point S'_1 to the circles S'_2, S'_3, and S'_4—which we will denote by t'^2_{12}, t'^2_{13}, and t'^2_{14}—and k is the power of the inversion. Hence

$$t'_{23}\frac{k}{t'_{12}t'_{13}} + t'_{34}\frac{k}{t'_{13}t'_{14}} = t'_{24}\frac{k}{t'_{12}t'_{14}}.$$

Dividing by k and eliminating fractions we obtain the equality $t'_{23}t'_{14} + t'_{34}t'_{12} = t'_{24}t'_{13}$, which is the relation we had to prove; this is so because, in view of property C of dilatation, $t'_{12} = t_{12}$, $t'_{13} = t_{13}$, $t'_{14} = t_{14}$, $t'_{23} = t_{23}$, $t'_{24} = t_{24}$, and $t'_{34} = t_{34}$.

Proof of sufficiency. Conversely, suppose we have, say, the relation

$$t_{12}t_{34} - t_{13}t_{24} + t_{14}t_{23} = 0.$$

As before, we have

$$t''_{23} = t'_{23}\frac{k}{t'_{12}t'_{13}} = t_{23}\frac{k}{t_{12}t_{13}}, \quad t''_{34} = t_{34}\frac{k}{t_{13}t_{14}}, \quad t''_{24} = t_{24}\frac{k}{t_{12}t_{14}}.$$

Dividing both sides of our relation $t_{12}t_{34} + t_{14}t_{23} = t_{13}t_{24}$ by $t_{12}t_{13}t_{14}$ and multiplying by k we obtain

$$t_{34}\frac{k}{t_{13}t_{14}} + t_{23}\frac{k}{t_{12}t_{13}} = t_{24}\frac{k}{t_{12}t_{14}}, \quad \text{or} \quad t''_{34} + t''_{23} = t''_{24}.$$

Now assume that the circles S''_2, S''_3, and S''_4 are not tangent to a single circle. We rotate S''_3 about the center of the circle S''_2 into the position of $\overline{S}''_3$, so that $\overline{S}''_3$ is tangent to the common tangent AC of the circles S''_2 and S''_4 (see Figure 205).

Then, clearly,

$$\overline{t}''_{34} + \overline{t}''_{23} = t''_{24},$$

where $\overline{t}''_{34}$ and $\overline{t}''_{23}$ are the respective tangent distances of the circles $\overline{S}''_3$ and $\overline{S}''_4$, and $\overline{S}''_3$ and $\overline{S}''_2$. But $\overline{t}''_{23} = t''_{23}$ (for $\overline{S}''_3$ is obtained from S''_3 by a rotation about the center of S''_2); hence $\overline{t}''_{34} = t''_{24} - \overline{t}''_{23} = t''_{24} - t''_{23} = t''_{34}$, which is impossible if $\overline{S}''_3$ is different from S''_3 (the tangent distances of the circles $\overline{S}''_3$ and S''_4 and the circles S''_3 and S''_4, where $\overline{S}''_3$ and S''_3 are congruent,

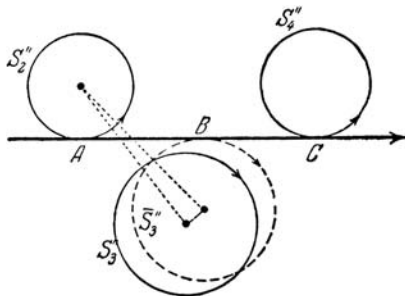


FIGURE 205

coincide only if $\overline{S}_3''$ is obtained from S_3'' by a rotation about the center of S_4''). We must therefore conclude that S_2', S_3', and S_4' are tangent to a single circle or line Σ' passing through S_1', and hence that S_1, S_2, S_3, and S_4 are tangent to a single circle (or line) Σ, or pass through a single point Σ.

Remark. We note the following simple proof of the theorem in problem 62(a) which does not make use of inversions. If the four circles S_1, S_2, S_3, and S_4 are tangent to a single line, then the proposition in problem 50 can be proved very simply (see the beginning of the proof of problem 50). If the four circles S_1, S_2, S_3, and S_4 pass through a single point, then the proof is also simple (see the solution of problem 51). If the four circles S_1, S_2, S_3, and S_4 are tangent to a single circle Σ, then we take Σ by a dilatation to a point Σ'; then the circles S_1, S_2, S_3, and S_4 go over to circles S_1', S_2', S_3', and S_4' which pass through the point Σ'. By property C of a dilatation the theorem in problem 50 is reduced to the theorem in problem 51, which can be proved without the use of inversion (and whose proof is much simpler than the proof of problem 50, which is based on the complicated property D of inversion).

63. First of all, we prove that the circle Σ, which passes through the midpoints D, E, and F of the sides AB, AC, and CB of the triangle ABC (the nine point circle of the triangle ABC) is tangent to the incircle s of that triangle. In fact, let a, b, and c be the lengths of the sides of the triangle ABC ($a \geq b \geq c$), and let P, Q, and R be the points of tangency of s and the sides AB, AC, and CB (Figure 206a). Then the length of the segment of the tangent drawn from D to s is $DP = AD - AP$. But

$$\begin{aligned} AP &= \frac{1}{2}(AP + AQ) = \frac{1}{2}(AB - BP + AC - CQ) \\ &= \frac{1}{2}(AB + AC - BR - CR) = \frac{c+b-a}{2}, \end{aligned}$$

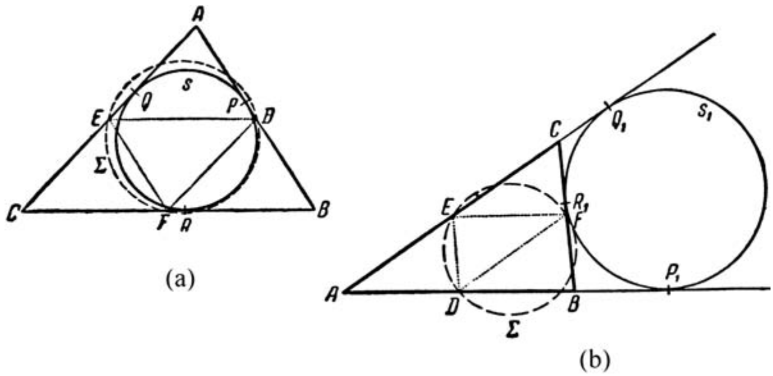


FIGURE 206

whence

$$DP = AD - AP = \frac{c}{2} - \frac{c+b-a}{2} = \frac{a-b}{2}.$$

In much the same way we prove that the lengths of the segments of the tangents drawn from the points E and F to the circle s are $\frac{a-c}{2}$ and $\frac{b-c}{2}$.

Now we take the points D, E, F, and the circle s as the circles S_1, S_2, S_3, and S_4 in the previous problem (that three of the four circles are points, that is, circles of zero radius, is, of course, not significant; the proof of the theorem in problem 62 carries over without change to the case when some of the circles S_1, S_2, S_3, and S_4 are replaced by points). Then we have:

$$t_{12} = DE = \frac{a}{2}, \quad t_{13} = DF = \frac{b}{2}, \quad t_{23} = EF = \frac{c}{2},$$
$$t_{14} = DP = \frac{a-b}{2}, \quad t_{24} = EQ = \frac{a-c}{2}, \quad t_{34} = FR = \frac{b-c}{2}.$$

From this it is clear that the relation in the previous problem holds; in fact,

$$t_{14}t_{23} + t_{12}t_{34} - t_{13}t_{24} = \frac{c}{2}\frac{a-b}{2} + \frac{a}{2}\frac{b-c}{2} - \frac{b}{2}\frac{a-c}{2} = 0.$$

Hence the "circles" D, E, and F and the circle s are tangent to the single circle Σ, which in our case means that the circle Σ, which passes through the points D, E, and F, is tangent to s.

In much the same way we show that the nine point circle is tangent to the excircles s_1, s_2, and s_3 of the triangle. In fact, let s_1, say, be the circle escribed in the angle A of the triangle, and let P_1, Q_1, and R_1 be its points of tangency with the sides of the triangle (Figure 206b). Then the lengths of

the segments of the tangents from D, E, and F to s_1 are

$$\begin{aligned} DP_1 &= AP_1 - AD = \frac{1}{2}(AP_1 + AQ_1) - AD \\ &= \frac{1}{2}(AB + BP_1 + AC + CQ_1) - AD \\ &= \frac{1}{2}(AB + AC + BR_1 + CR_1) - AD \\ &= \frac{1}{2}(AB + AC + BC) - AD = \frac{a+b+c}{2} - \frac{c}{2} = \frac{a+b}{2}, \\ EQ_1 &= \frac{a+c}{2}, \end{aligned}$$

and

$$\begin{aligned} FR_1 &= BR_1 - BF = BP_1 - BF = (AP_1 - AB) - BF \\ &= \left(\frac{a+b+c}{2} - c\right) - \frac{a}{2} = \frac{b-c}{2}; \end{aligned}$$

and therefore

$$DE \cdot FR_1 - DF \cdot EQ_1 + EF \cdot DP_1 = \frac{c}{2}\frac{a+b}{2} + \frac{a}{2}\frac{b-c}{2} - \frac{b}{2}\frac{a+c}{2} = 0,$$

which proves that the circle Σ is tangent to s_1.

Remark. Using the theorem in problem 62 we can show at once that the circles s, s_1, s_2, and s_3 are tangent to a single circle Σ (moreover, Σ has tangency of the same kind with s_1, s_2, and s_3 and tangency of the opposite kind with s, that is, Σ is outer-tangent to s_1, s_2, and s_3 and inner-tangent to s, or it is inner-tangent to s_1, s_2, and s_3 and outer-tangent to s; it is not difficult to prove that only the first of these possibilities obtains). In fact, let us denote the tangent distances of the circles s and s_1, s and s_2, and s and s_3, by t_{01}, t_{02}, and t_{03}, and the tangent distances of the circles s_1, s_2, and s_3 taken two at a time by t_{12}, t_{13}, and t_{23}. Then, clearly,

$$t_{01} = RR_1 = BR_1 - BR = \frac{a+b-c}{2} - \frac{a-b+c}{2} = b - c,$$

and similarly

$$t_{02} = a - c, \quad t_{03} = a - b;$$

and in much the same way, if P_3 is the point of tangency of the circle s_3 with the side AB, then

$$t_{13} = P_1P_3 = AP_1 + BP_3 - AB = \frac{a+b+c}{2} + \frac{b+c-a}{2} - c = b,$$

and, similarly

$$t_{12} = c, \quad t_{23} = a.$$

From this we see that

$$t_{01}t_{23} - t_{02}t_{13} + t_{03}t_{12} = a(b-c) - b(a-c) + c(a-b) = 0,$$

which implies that s, s_1, s_2, and s_3 are tangent to a single circle Σ. However, to prove that Σ coincides with the nine point circle of the triangle we must follow a different path, namely, the one we followed when solving the present problem.

64. Assume for definiteness that the circles S_1, S_2, S_3, and S_4 are tangent to Σ_1, Σ_2, and Σ_3 as shown in Figure 207. Then, denoting the segment of the common outer tangent of the circles S_1 and S_2 by t_{12} and the segment of their common inner tangent by $\bar{t}_{12}$, and by introducing analogous notations for the other pairs of circles, we obtain, using the theorem in problem 62(a), the following three equalities: the equality

$$\bar{t}_{13}t_{24} = \bar{t}_{12}t_{34} + \bar{t}_{14}t_{23},$$

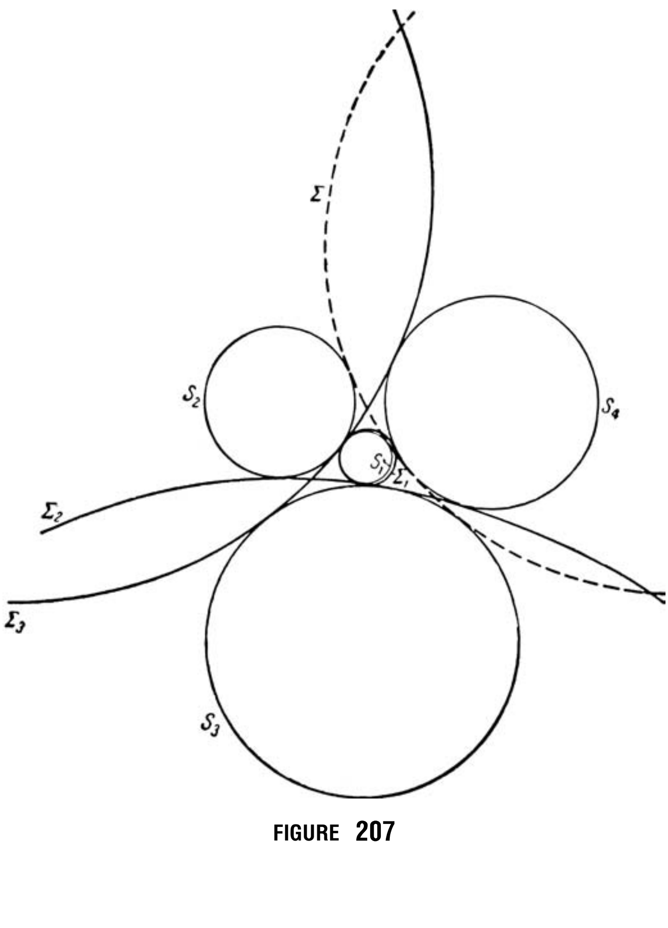


FIGURE 207

implied by the fact that S_1, S_2, S_3, and S_4 are tangent to the single circle Σ_1; the equality

$$t_{12}\bar{t}_{34} = \bar{t}_{13}t_{24} + t_{14}\bar{t}_{23},$$

implied by the fact that S_1, S_2, S_3, and S_4 are tangent to the single circle Σ_2; and the equality

$$t_{13}\bar{t}_{24} = \bar{t}_{12}t_{34} + t_{14}\bar{t}_{23},$$

implied by the fact that S_1, S_2, S_3, and S_4 are tangent to the single circle Σ_3.

Adding the first two of these equalities and subtracting the third we obtain

$$\bar{t}_{13}t_{24} + t_{12}\bar{t}_{34} - t_{13}\bar{t}_{24} = \bar{t}_{14}t_{23} + \bar{t}_{13}t_{24},$$

that is,

$$t_{12}\bar{t}_{34} = t_{13}\bar{t}_{24} + \bar{t}_{14}t_{23}.$$

By the theorem in problem 62(b), this implies that S_1, S_2, S_3, and S_4 are tangent to some circle Σ (which has like tangency with S_1, S_2, and S_3 and unlike tangency with S_4).

65. (a) We view the circles S_1, S_2, and S_3 as directed and use a specially chosen axial inversion to take them to points S_1', S_2', and S_3' (see p. 113). Here the circle Σ, tangent to S_1, S_2, and S_3, goes over to a circle Σ' passing through the points S_1', S_2', and S_3'. After constructing Σ', we obtain the circle Σ, the preimage of Σ' under our axial inversion. Since we can assign to Σ two opposite directions, the problem has two solutions for each assignment of directions to S_1, S_2, and S_3; the problem can have up to eight solutions (compare this with the solution of problem 61).

(b) We use an axial inversion to take the circles S_1, S_2, and S_3 (which we consider directed; see the solution of problem 62) to three points S_1', S_2', and S_3' (see p. 113); the circle S_4 goes over to a new circle S_4' tangent to the circle Σ', passing through the points S_1', S_2', and S_3' (Figure 208). The tangent distances t_{12}, t_{13}, and t_{23} go over to the segments $S_1'S_2'$, $S_1'S_3'$, and $S_2'S_3'$, and the tangent distances t_{14}, t_{24}, and t_{34} go over to the lengths of the tangents drawn from the points S_1', S_2', and S_3' to the circle S_4'.

Let A be the point of tangency of the circles S_4' and Σ' and let S_1'', S_2'', and S_3'' be the points of intersection of the lines AS_1', AS_2', and AS_3' with the circle S_4'. The circles Σ' and S_4' are centrally similar with center of

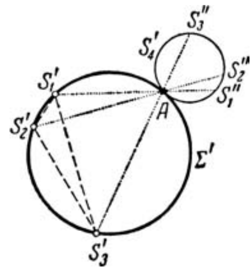


FIGURE 208

similarity at A; hence

$$AS_1'' = \frac{r_4'}{r'}AS_1', \quad AS_2'' = \frac{r_4'}{r'}AS_2', \quad AS_3'' = \frac{r_4'}{r'}AS_3',$$

where r_4' and r' are the radii of the circles S_4' and Σ'. Clearly,

$$t_{14}^2 = S_1'A \cdot S_1'S_1''$$

whence

$$t_{14} = \sqrt{AS_1'(AS_1' + AS_1'')} = \sqrt{AS_1'\left(AS_1' + \frac{r_4'}{r'}AS_1'\right)} = AS_1'\sqrt{1 + \frac{r_4'}{r'}},$$

and similarly

$$t_{24} = AS_2'\sqrt{1 + \frac{r_4'}{r'}}, \quad t_{34} = AS_3'\sqrt{1 + \frac{r_4'}{r'}}.$$

The points A, S_1', S_2', and S_3' lie on the circle Σ'. Hence, using the theorem of Ptolemy,[3] we have

$$AS_1' \cdot S_2'S_3' + AS_3' \cdot S_1'S_2' = AS_2' \cdot S_1'S_3'$$

(here we are assuming that the points A, S_1', S_2', and S_3' are disposed on the circle Σ' in the order in which they appear in Figure 208). Multiplying the last equality by $\sqrt{1 + \frac{r_4'}{r'}}$, changing $S_1'S_2'$ to t_{12}, and so on, $\sqrt{1 + \frac{r_4'}{r'}}AS_1'$ to t_{14}, and so on (see property C of axial inversion), we obtain the required relation

$$t_{12}t_{34} + t_{14}t_{23} = t_{13}t_{24}.$$

66. This problem is close to problem 12 in Section 1 (it is its dual). Let $ABCDEF$ be a hexagon circumscribed about the circle S and let S_1 be an arbitrary circle tangent to the sides AB and DE of the hexagon (Figure

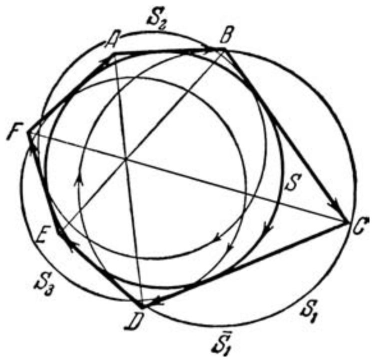


FIGURE 209

209; we suppose the circles and lines directed. Moreover, the direction of S can be chosen arbitrarily).

The axial inversion with central line BE and directing circle S takes the circle S_1 to a circle S_2 tangent to BC and EF (for AB goes over to BC and DE to EF); another axial inversion, with central line CF and directing circle S, takes the circle S_2 to a circle S_3 tangent to the lines CD and FA; finally, the axial inversion with central line DA and directing circle S takes the circle S_3 to a circle $\overline{S}_1$ tangent to the lines DE and AB.

By property C of axial inversion, the circles S_1, S_2, S_3, and $\overline{S}_1$ have the same tangent distances with the circle S (which all the axial inversions under consideration take to itself). But the fact that the circles S_1 and $\overline{S}_1$ are tangent to the lines DE and AB and have the same tangent distances with the circle S implies that they are coincident.[4]

Thus we see that the lines BE, CF, and DA are the central lines of axial inversions which take S_1 to S_2, S_2 to S_3, and S_3 to S_1. But this implies that these lines are the radical axes of those three circles taken two at a time (see the proof of property C of axial inversion). But then these three lines must meet in a point, the radical center of the circles S_1, S_2, and S_3 (see p. 53), which is what we had to prove.

67. (a) By an axial inversion we take the circles S_1 and S_2 with common tangents a_1 and a_2 to points S_1' and S_2' (see p. 113; we suppose all circles and lines to be directed). Here Figure 107a will go over to Figure 210a; we must show that if the lines a_1', b_1', c_1' and d_1' are tangent to the same circle Σ', then the lines a_2', b_2', c_2', and d_2' are tangent to the same circle $\overline{\Sigma}'$.

Let us denote the points of intersection of the lines under consideration and their points of tangency with the circles S_3' and S_4' as in Figure 210a.

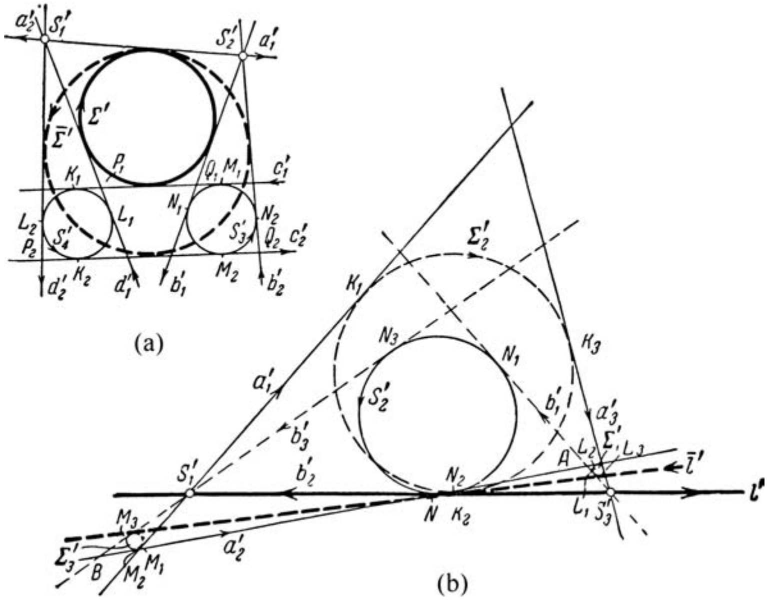


FIGURE 210

Since the circle Σ' is inscribed in the quadrilateral $S_1'P_1Q_1S_2'$, it follows that $S_1'S_2' + P_1Q_1 = S_1'P_1 + S_2'Q_1$. If we add to the left side of the last equality the segments P_1K_1 and Q_1M_1 and to the right side the segments P_1L_1 and Q_1N_1 equal to the latter, then we obtain the equality

$$S_1'S_2' + K_1M_1 = S_1'L_1 + S_2'N_1,$$

or

$$S_1'S_2' + K_2M_2 = S_1'L_2 + S_2'N_2$$

(for $K_1M_1 = K_2M_2$, $S_1'L_1 = S_1'L_2$, $S_2'N_1 = S_2'N_2$). If we add to the left side of the last equality the segments K_2P_2 and M_2Q_2 and to the right side the segments L_2P_2 and N_2Q_2 equal to the latter, then we obtain the equality

$$S_1'S_2' + P_2Q_2 = S_1'P_2 + S_2'Q_2,$$

which implies that we can inscribe the circle Σ' in the quadrilateral $S_1'P_2Q_2S_2'$.[5]

(b) We apply an axial inversion which takes the circles S_1 and S_3 with common tangents l and l_2 to points S_1' and S_3'; then Figure 107b will go

over to Figure 210b (see p. 113; all circles and lines are supposed directed). We are to show that the circles Σ_1', Σ_2' and Σ_3' in Figure 210b are tangent to a single line $\overline{l}'$, that is

$$t_{12} + t_{23} = t_{13},$$

where t_{12} is the tangent distance of Σ_1' and Σ_2', and so on (see the solution of problem 62(b)). But from Figure 210b we have:

$$t_{12} = K_3L_3 = S_3'K_2 - S_3'L_1, \quad t_{23} = K_1M_1 = S_1'K_2 + S_1'M_3,$$
$$t_{13} = M_2L_2 = AB - BM_2 + AL_2 = AB - BM_3 + AL_1,$$

whence

$$\begin{aligned} &t_{12} + t_{23} - t_{13} \\ &\quad = (S_3'K_2 + S_1'K_2) - (S_3'L_1 + AL_1) + (S_1'M_3 + BM_3) - AB \\ &\quad = S_3'S_1' - S_3'A + S_1'B - AB \\ &\quad = S_3'S_1' - S_3'A - (AN_2 + BN_2) + S_1'B \\ &\quad = S_3'S_1' - (S_3'A + AN_1) - (BN_3 - S_1'B) \\ &\quad = S_3'S_1' - S_3'N_1 - S_1'N_3 \\ &\quad = S_3'S_1' - S_3'N - S_1'N = 0, \end{aligned}$$

which was to be proved.[6]

68. We consider separately three cases.

1°. The axis of similarity of the circles A, B, and C (which we suppose oriented) does not intersect any of them. Then all three circles can be taken by an axial inversion to three points A', B', and C'. Then the figure associated with the problem goes over to Figure 212a. The tangent distance x is equal to the length of the segment $C'D'$, that is, the length of the median of the triangle $A'B'C'$; by a well-known formula

$$x^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2).$$

2°. We apply an axial inversion which takes the circles A and B to points A' and B'; suppose that the circle C goes over to a circle C' which intersects $A'B'$ at points M and N (if C' did not intersect $A'B'$ then we would be dealing with the first case). By an axial inversion with central line $A'B'$ we can take the circle C' to a circle C'' with diameter MN (see p. 111); then the figure of the problem goes over to Figure 212b and the problem is reduced to the determination of the length of the segment $D'P$

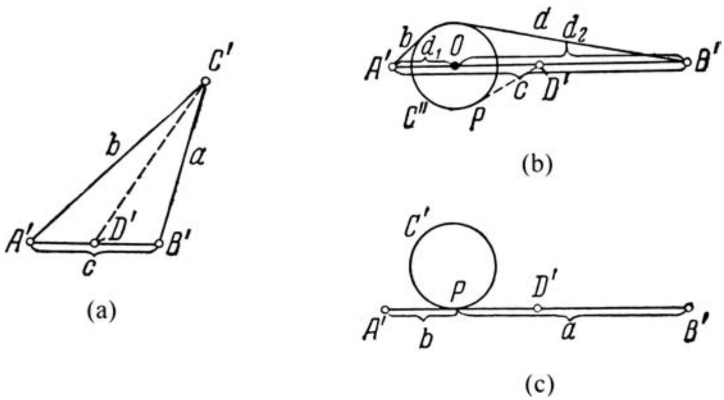


FIGURE 212

of the tangent drawn from the midpoint D' of the segment $A'B'$ to the circle C''. Let the distances $A'O$ and $B'O$ (O is the center of C'') be d_1 and d_2 and let the radius of C'' be r; then

$$d_1^2 - r^2 = b^2 \qquad \text{and} \qquad d_2^2 - r^2 = a^2,$$

whence

$$d_1^2 - d_2^2 = b^2 - a^2.$$

Now, depending on whether O lies between A' and B' (Figure 212b) or outside $A'B'$, we have:[7]

$$d_1 \pm d_2 = c, \quad d_1 \mp d_2 = \frac{b^2 - a^2}{c},$$

and in both cases

$$d_1 = \frac{b^2 + c^2 - a^2}{2c}, \quad r^2 = d_1^2 - b^2 = \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2 - b^2$$
$$= \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{4c^2}.$$

Since $D'O = A'D - A'O = \frac{c}{2} - \frac{b^2+c^2-a^2}{2c} = \frac{a^2-b^2}{2c}$, we end up with the

same formula:

$$\begin{aligned} x^2 &= D'P^2 = D'O^2 - r^2 \\ &= \left(\frac{a^2-b^2}{2c}\right)^2 - \frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2}{4c^2} \\ &= \frac{1}{4}(2a^2+2b^2-c^2). \end{aligned}$$

3°. We take A and B by an axial inversion to points A' and B'; let the image of the circle C be a circle C' tangent to $A'B'$ at the point P (Figure 212c). The problem reduces to the determination of the length of the segment $D'P$ of the tangent drawn from the midpoint D' of the segment $A'B'$ to the circle C'. Clearly, $A'P = b$, $B'P = a$, and, for example, in the case of Figure 212c, we have:[7]

$$x = D'P = A'D' - A'P = \frac{c}{2} - b.$$

[We note that the formula $x^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2)$ applies to this case as well: in fact, here $c = a + b$, and therefore $\frac{1}{4}(2a^2 + 2b^2 - c^2) = \frac{1}{4}[2a^2 + 2b^2 - (a+b)^2] = \frac{1}{4}(a-b)^2 = \frac{1}{4}[(c-b)-b]^2 = (\frac{c}{2} - b)^2$.]

Remark. Case 1° is characterized by the fact that the largest of the segments a, b, and c is smaller than the sum of the other two (it is possible to construct a triangle from a, b, and c; see Figure 212a); case 2°—by the fact that the largest of the segments a, b, and c is greater than the sum of the other two; and case 3°—by the fact that the largest of the segments a, b, and c is equal to the sum of the other two.

69. We consider separately the three cases with which we dealt in the solution of the previous problem.

1°. We take the three circles A, B, and C by an axial inversion to three points A', B', and C'. Then Figure 19 goes over to Figure 213a and all assertions stated in the problem follow from the properties of the medians in a triangle.

2°. By two axial inversions we can take the circles A, B, and C to two points A', B', and a circle C'' whose center O lies on the line $A'B'$; then Figure 109 goes over to Figure 213b. Let M_1, M_2, and M_3 be three circles, tangent to the tangents drawn from A' to E'', from B' to F'', and from D' to C'', which divide these tangents in the ratio 2:1 (beginning with A', B', and C''); we must show that these three circles are coincident.

We denote the centers of E'', F'', M_1, M_2, and M_3 by $\overline{O}$, o, O_1, O_2, and O_3 and their radii by $\overline{r}$, r, r_1, r_2, and r_3; further, we put $A'O = d_1$,

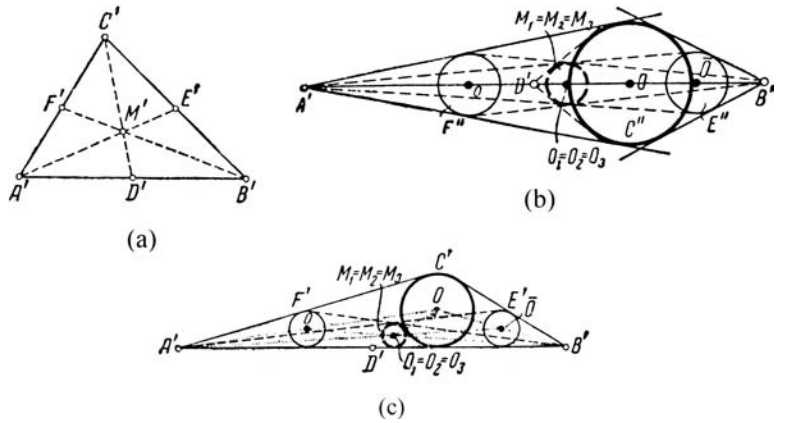


FIGURE 213

$B'O = d_2$,[8] and $A'B' = c$, and denote the radius of C'' by R. The circles E'' and C'' are centrally similar with center of similarity B' and coefficient of similarity $\frac{1}{2}$; hence $B'\overline{O} = \frac{d_2}{2}$ and $\overline{r} = \frac{R}{2}$. The circles E'' and M_1 are centrally similar with center of similarity A' and coefficient of similarity $\frac{2}{3}$; hence

$$A'O_1 = \frac{2}{3}A'\overline{O} = \frac{2}{3}\left(c - \frac{d_2}{2}\right) = \frac{2}{3}c - \frac{d_2}{3}, \quad r_1 = \frac{2}{3}\overline{r} = \frac{1}{3}R.$$

In much the same way we obtain:

$$B'O_2 = \frac{2}{3}c - \frac{d_1}{3}, \quad r_2 = \frac{1}{3}R;$$

this already implies the coincidence of the circles M_1 and M_2 (for their radii are equal and their centers coincide because $A'O_1 + B'O_2 = \frac{4}{3}c - \frac{d_1+d_2}{3} = c = A'B'$). Finally, the circles M_3 and C'' are centrally similar with center of similarity D' and coefficient of similarity $\frac{1}{3}$; hence

$$r_3 = \frac{1}{3}R, \quad D'O_3 = \frac{1}{3}D'O = \frac{1}{3}\left(d_1 - \frac{c}{2}\right) = \frac{d_1}{3} - \frac{c}{6},$$

which means that M_3 coincides with M_1 and M_2 (for their radii are equal and their centers coincide because $A'O_3 = A'D' + D'O_3 = \frac{c}{2} + \frac{c-d_2}{3} - \frac{c}{6} = \frac{2}{3}c - \frac{d_2}{3}$).

3°. The circles A, B, and C can be taken by an axial inversion to points A', B', and a circle C' tangent to the line $A'B'$; then Figure 109 goes over to Figure 213c; we define the circles M_1, M_2, and M_3 as before; we must prove that they coincide; we keep the previous notations for the centers and

radii of the circles E', F', M_1, M_2, and M_3. Just as in 2°, we show that $r_1 = r_2 = r_3 = \frac{R}{3}$; thus we need only show that the centers of these circles coincide. But, clearly, the points O_1, O_2, and O_3 divide the medians $A'\overline{O}$, $B'o$, and OD' of the triangle $A'B'O$ in the ratio 2:1; hence these three points coincide with the center of the triangle $A'B'O$.

70. If o is the central line of the axial inversion, then $\widehat{ac} = \widehat{ao} - \widehat{co}$, and so on (see Figure 214; the signs of the angles $\widehat{ac}$, and so on, are based on the convention stated on p. 106). Hence

$$\sin\frac{\widehat{ac}}{2} = \sin\frac{\widehat{ao}-\widehat{co}}{2} = \sin\frac{\widehat{ao}}{2}\cos\frac{\widehat{co}}{2} - \sin\frac{\widehat{co}}{2}\cos\frac{\widehat{ao}}{2}$$
$$= \cos\frac{\widehat{ao}}{2}\cos\frac{\widehat{co}}{2}\left(\tan\frac{\widehat{ao}}{2} - \tan\frac{\widehat{co}}{2}\right), \quad \text{and so on.}$$

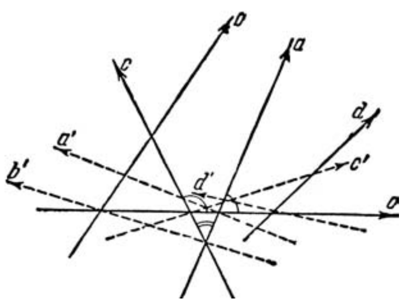


FIGURE 214

This implies that

$$\frac{\sin\frac{\widehat{ac}}{2}}{\sin\frac{\widehat{bc}}{2}} : \frac{\sin\frac{\widehat{ad}}{2}}{\sin\frac{\widehat{bd}}{2}} = \frac{\tan\frac{\widehat{ao}}{2} - \tan\frac{\widehat{co}}{2}}{\tan\frac{\widehat{bo}}{2} - \tan\frac{\widehat{co}}{2}} : \frac{\tan\frac{\widehat{ao}}{2} - \tan\frac{\widehat{do}}{2}}{\tan\frac{\widehat{bo}}{2} - \tan\frac{\widehat{do}}{2}}.$$

Similarly,

$$\frac{\sin\frac{\widehat{a'c'}}{2}}{\sin\frac{\widehat{b'c'}}{2}} : \frac{\sin\frac{\widehat{a'd'}}{2}}{\sin\frac{\widehat{b'd'}}{2}} = \frac{\tan\frac{\widehat{a'o}}{2} - \tan\frac{\widehat{c'o}}{2}}{\tan\frac{\widehat{b'o}}{2} - \tan\frac{\widehat{c'o}}{2}} : \frac{\tan\frac{\widehat{a'o}}{2} - \tan\frac{\widehat{d'o}}{2}}{\tan\frac{\widehat{b'o}}{2} - \tan\frac{\widehat{d'o}}{2}}.$$

But, by the definition of an axial inversion, $\tan\frac{\widehat{a'o}}{2} = \frac{k}{\tan\frac{\widehat{ao}}{2}}$, and so on. Substituting these expressions in the last relation above, we easily obtain

the required relation

$$\frac{\sin\frac{\widehat{a'c'}}{2}}{\sin\frac{\widehat{b'c'}}{2}} : \frac{\sin\frac{\widehat{a'd'}}{2}}{\sin\frac{\widehat{b'd'}}{2}} = \frac{\sin\frac{\widehat{ac}}{2}}{\sin\frac{\widehat{bc}}{2}} : \frac{\sin\frac{\widehat{ad}}{2}}{\sin\frac{\widehat{bd}}{2}}.$$

71. If the sequence $I_1, I_2, I_3, \ldots, I_n$ of axial inversions takes two different points A and B again to points A' and B', then it takes all points of the line AB to points. In fact, in that case, two directed lines l_1 and l_2 passing through A and B and differing only by direction go over to lines l_1' and l_2' passing through A' and B' and differing only by direction; since no circle with nonzero radius can be tangent to both l_1' and l_2', it follows that all points of the line AB go over to points of the line $A'B'$ (see pp. 111–113). Hence if a sequence of axial inversions takes three noncollinear points A, B, and C to points, then it takes all points of the plane to points; in fact, if M is an arbitrary point and N is the point of intersection of AM and BC (Figure 215), then N goes over to a point because it lies on the line BC, and M goes over to a point because it lies on the line AN. It follows that there arise the following four cases:

The sequence of axial inversions:

1° does not take any point of the plane to a point;

2° takes just the point O to a point (we will show below that this is impossible);

3° takes to points the points of some line o (and no others);

4° takes all points of the plane to points.

The aim of the rest of the problem is to see which of these cases is taking place, and, if it is the case 3° (the most relevant case; it turns out that 1° and 4° are ruled out), to find the line o, the required locus.

We consider an arbitrary line AB in the plane. Two (directed) lines l_1 and l_2 passing through the points A and B and differing only by direction go over to two new lines l_1' and l_2'. We analyze separately all possibilities.

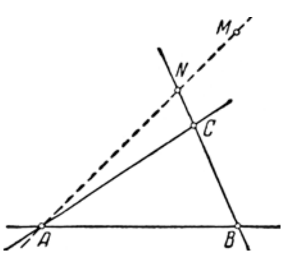


FIGURE 215

(1) The lines l_1' and l_2' also differ only by direction. Then all points of the line AB go over to points of the lines l_1' and l_2' which coincide in position in the plane. Therefore, the only possiblities that arise are 3° (in which case o coincides with AB) or 4°. To clarify which of these two cases obtains, it suffices to check whether an arbitrary point C, not on the line AB, goes over to a circle or a point.

(2) The directed lines l_1' and l_2' are parallel. This possibility can be immediately rejected because the points of the line AB must go over to circles tangent to l_1' and l_2', but (directed) circles tangent to parallel lines don't exist.

(3) The lines l_1' and l_2' are counterparallel; the points of the line AB go over to circles with the same radius tangent to l_1' and l_2' (Figure 216). Let CD be an arbitrary line parallel to (but different from) AB; lines m_1 and m_2 passing through the points C and D and differing only by direction go over to lines m_1' and m_2'.

We consider various possibilities.

a. If m_1' and m_2' differ only by direction, then we are back to possibility (1); here 3° obtains and o coincides with CD.

b. The lines m_1' and m_2' cannot be parallel (see (2) above).

c. If m_1' and m_2' are counterparallel, then the points of the line CD go over to circles with equal radii tangent to m_1' and m_2'. Moreover, the line of centers of these circles must be parallel to the line of centers of the circles which are the images of the points of AB; otherwise we could find a point of the line AB and a point of the line CD whose images would be concentric circles—a possibility which contradicts the fact that the tangent distance between these points must be equal to the distance between the initial points (and so must exist). Further, if the radii of the latter circles are equal (in terms of magnitude and sign) to the radii of the circles that are the images of the points of the line AB (Figure 216a), then the points of any line MN, where M is a point of AB and N is a point of CD, go over to circles that have no common tangents with the circles M_1' and N_2' with equal radii; hence all points in the plane go over to circles with the same radius, and thus case 1° obtains. On the other hand, if the points of the line AB and the points of the line CD go over to circles with different radii (Figure 216b), then the points of any line MN, where M is a point of AB and N is a point of CD, go over to circles which have common tangents with the circles M' and N' with different radii; it follows that a point P of the line MN goes over to a point P'; the position of P is determined by the

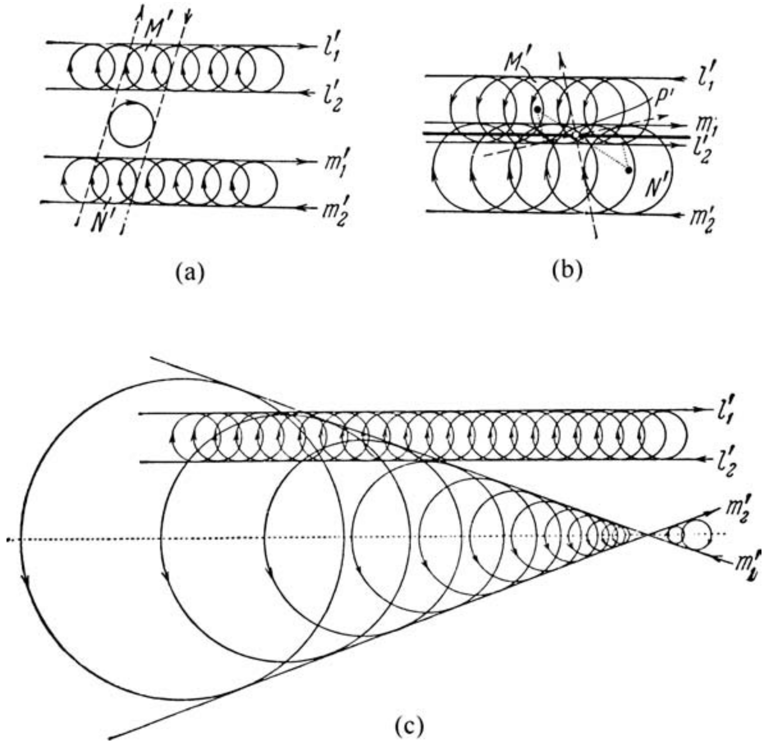


FIGURE 216

fact that

$$\frac{MP}{NP} = \frac{\text{the tangent distance between the circle } M' \text{ and the point } P'}{\text{the tangent distance between the circle } N' \text{ and the point } P'}$$

$$= \frac{\text{radius of } M'}{\text{radius of } N'}$$

(Figure 216b). The locus of all such points P is the line o, parallel to AB and CD; thus we are again back to the case 3°.

d. Finally, it is easy to show that the lines m_1' and m_2' cannot intersect one another. Suppose they do, and suppose that points of the line CD go over to circles tangent to the intersecting lines m_1' and m_2'. Each of these circles must have a definite tangent distance with each of the circles tangent to l_1' and l_2' (this tangent distance is equal to the distance between the corresponding points). Therefore the line of centers of all these circles (the bisector of the angle formed by m_1' and m_2') must be parallel to l_1' and l_2'—if

this were not so, then there would be among those circles a circle concentric with one of the circles tangent to l'_1 and l'_2. But also in this, the latter, case, it would be possible to find two circles, one tangent to m'_1 and m'_2 and the other tangent to l'_1 and l'_2, with no tangent distance—for proof it is enough to note that circles tangent to m'_1 and m'_2 can be arbitrarily large, and therefore some of them include in their interiors circles tangent to l'_1 and l'_2 (Figure 216c).

(4) The lines l'_1 and l'_2 intersect one another. In that case, a definite point P of the line AB goes over to the point P' of intersection of l'_1 and l'_2; it is easy to find P because its distance from A is equal to the tangent distance of the point P' and the circle A' (which is the image of A). Then we consider any other line CD not passing through the point P; the lines m_1 and m_2, passing through C and D, will go over to the intersecting lines m'_1 and m'_2 (for the preceding analysis shows that if the lines l'_1 and l'_2 intersect one another, then the lines m'_1 and m'_2 cannot differ just by their directions or be counterparallel). Thus it will be possible to find just one more point Q which goes over to the point Q'; hence here the case 3° obtains and o coincides with the line PQ.

72. This problem is close to problem 20 (it is its dual). Assume that our problem has been solved and $A_1A_2A_3 \ldots A_n$ is the required n-gon whose vertices lie on the given lines $l_1, l_2, l_3, \ldots, l_{n-1}, l_n$ (Figure 217). We suppose all lines and circles appearing in the solution of this problem directed; moreover, the directions of the circle S and the lines $l_1, l_2, l_3, \ldots, l_n$ can be chosen arbitrarily. We apply successively n axial inversions with central

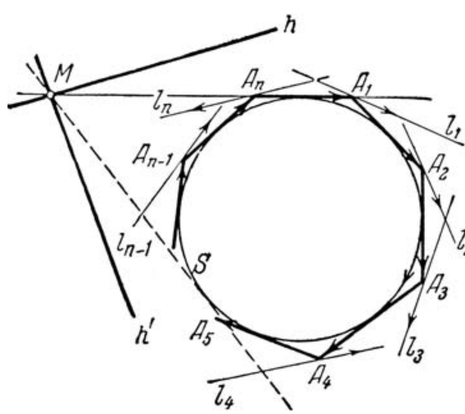


FIGURE 217

lines $l_1, l_2, l_3, \ldots, l_{n-1}, l_n$ and the same directing circle S. These inversions will take the side A_nA_1 successively to $A_1A_2, A_2A_3, \ldots,\ A_{n-1}A_n$ and finally again to A_nA_1. Thus we see that the totality of the n given axial inversions takes the line A_nA_1 to itself. Hence the problem is reduced to finding the tangent A_nA_1 to the circle S which is fixed by the given sequence of axial inversions. We can continue along different paths.

First solution (close to the solution of problem 20). All points which our sequence of axial inversions takes to points lie on a single line h whose image is a line h' (see problem 71).[9] Let M be the point of intersection of h and A_1A_n. The n axial inversions take M to a point of the line h'; since A_1A_n goes over to itself, the point in question will be the point M' of intersection of h' and A_1A_n. Now, as in the solution of problem 20, we must consider separately two cases.

1°. n is even. In this case the tangent distance of M and S is equal to the tangent distance of M' and S in magnitude and direction (see the text in fine print on p. 13). Hence M coincides with M'; therefore this is the point of intersection of h and h'. By drawing the tangent from M to S we will obtain the side A_1A_n of the required n-gon. If M lies outside S, the problem has two solutions; if M lies on S, the problem has one solution; if M lies inside S or $h \parallel h'$, there is no solution. In the special case when h coincides with h' the problem is undetermined.

2°. n is odd. In this case the tangent distance of M and S is equal to the tangent distance of M' and S in magnitude but is opposite to it in direction; hence the problem is reduced to finding a tangent A_nA_1 to S whose segment between the known lines h and h' is halved by the point of tangency of A_nA_1 and S (Figure 218). In general, this is a very difficult problem, but in our case it is greatly simplified by the fact that *the lines h and h' are equidistant from the center O of the circle S*. In fact, if n is odd, then the image of M, the point of intersection of h and h', is not M: our n axial inversions take M to a point M_1 of the line h' and M is the image of a point M_2 of h. By property C of axial inversion, the (tangent) distances M_2M and MM_1 are equal; on the other hand, the tangent distance of M_2 and S is equal to the tangent distance of M_1 and S. It follows that M_2, M, and M_1 lie on a circle Σ concentric with S and the lines M_1M and MM_2 are equidistant from the center of S. Considerations of symmetry imply that the required line A_nA_1 is perpendicular to the bisector of the angle between h and h' (Figure 218). The problem has two solutions.

Second solution (close to the solution of problem 48). Let p, q, and r be three arbitrary (directed) lines which our sequence of n inversions takes

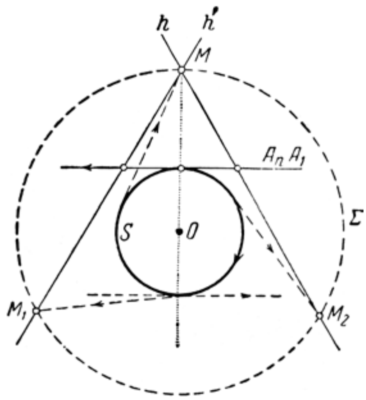


FIGURE 218

to three (directed) lines p', q', and r'; we denote the required line A_1A_n, which coincides with its image, by x. Since an axial inversion preserves the cross ratio of four lines (see problem 70), we have:

$$\frac{\sin\frac{\widehat{pr}}{2}}{\sin\frac{\widehat{qr}}{2}} : \frac{\sin\frac{\widehat{px}}{2}}{\sin\frac{\widehat{qx}}{2}} = \frac{\sin\frac{\widehat{p'r'}}{2}}{\sin\frac{\widehat{q'r'}}{2}} : \frac{\sin\frac{\widehat{p'x}}{2}}{\sin\frac{\widehat{q'x}}{2}}.$$

Now let o be an arbitrary (directed) line in the plane. The latter equality can be rewritten as follows:

$$\frac{\tan\frac{\widehat{po}}{2} - \tan\frac{\widehat{ro}}{2}}{\tan\frac{\widehat{qo}}{2} - \tan\frac{\widehat{ro}}{2}} : \frac{\tan\frac{\widehat{po}}{2} - \tan\frac{\widehat{xo}}{2}}{\tan\frac{\widehat{qo}}{2} - \tan\frac{\widehat{xo}}{2}}$$

$$= \frac{\tan\frac{\widehat{p'o}}{2} - \tan\frac{\widehat{r'o}}{2}}{\tan\frac{\widehat{q'o}}{2} - \tan\frac{\widehat{r'o}}{2}} : \frac{\tan\frac{\widehat{p'o}}{2} - \tan\frac{\widehat{xo}}{2}}{\tan\frac{\widehat{q'o}}{2} - \tan\frac{\widehat{xo}}{2}} \qquad (*)$$

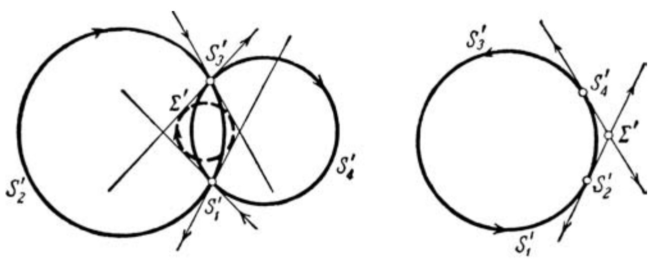
(see the solution of problem 70). But this equality is a quadratic equation in the unknown $\tan\frac{\widehat{xo}}{2}$. After obtaining $\tan\frac{\widehat{xo}}{2}$ we obtain the angle $\widehat{xo}$; knowing the direction of the (directed) line x tangent to the (directed) circle S we can easily obtain that line. Then it is a simple matter to construct the remaining sides of the required n-gon. The construction is carried out with ruler and compass.

Depending on the number of solutions of the quadratic equation $(*)$, the problem has two solutions, one solution, or no solution (we leave it to the reader to explain why the solution does not depend on the choice of directions of the circle S and the lines $l_1, l_2, \ldots, l_n$).

73. Guided by Theorem 2 (p. 120), we consider separately three cases.

1°. The circles S_1 and S_2 have two common tangents (we suppose all lines and circles directed). In that case we can take them by an axial inversion to two points S_1' and S_2'; then Figure 112 will go over to Figure 219a. The fact that the tangents to the circles S_1' and S_2' in that figure are tangent to a single circle Σ' follows from considerations of symmetry.

2°. The circles S_1 and S_2 have no common tangents. In that case we can take them by an axial inversion to circles S_1' and S_2' which differ only by direction; then Figure 112 will go over to Figure 219b (a circle of nonzero radius cannot be tangent to S_1' and S_2'). The tangents to S_1' and S_2' differ only by direction; they meet in the point Σ', the image under the axial inversion of the circle Σ tangent to the four common tangents to the initial circles S_1, S_2, S_3, and S_4 which pass through their points of tangency.



(a) (b)

FIGURE 219

3°. The case when the circles S_1 and S_2 are tangent to one another is of no interest, for then all four circles S_1, S_2, S_3, and S_4 must be tangent to one another in a single point and the four common tangents in the statement of the theorem must coincide. [If every two of three directed circles S_1, S_2, and S_3 are tangent to one another, then the three points of tangency must coincide—no other cases of pairwise tangency of three nondirected circles can satisfy the conditions of tangency of directed circles.]

74. (a) We suppose the circles S_1 and S_2 and the line l directed and consider separately three possible cases.

1°. The circles S_1 and S_2 have no common tangents. In that case we can take them by an axial inversion to circles S_1' and S_2' which differ only by direction (see Theorem 2 on p. 120); suppose that here the line l goes over to a line l'. The required circle Σ goes over to a circle Σ' tangent to S_1', S_2', and l' (or to a point Σ' common to S_1', S_2', and l'). But a circle with

nonzero radius cannot be simultaneously tangent to two circles which differ only by direction; hence Σ' is the point of intersection of S_1' and l'. After obtaining Σ' we can also construct the required circle Σ, which corresponds to the point Σ' under the required axial inversion.

The problem has two solutions, one solution, or no solution.

2°. The circles S_1 and S_2 have one common tangent (are tangent to one another). In that case, we can take them by an axial inversion to a point S_1' and a circle S_2' passing through it; here let l go over to some line l'. The required circle Σ will go over to a circle Σ' tangent to S_2' at the point S_1' and tangent to l'; it is not difficult to construct Σ' (its center lies on the line OS_1', where O is the center of S_2', and on the bisector of the angle formed by l' and the tangent to S_2' at the point S_1'). The problem can have up to two solutions; if l' is tangent to S_2' at S_1' (that is, S_1, S_2, and l are tangent in a single point), then the problem is undetermined.

3°. The circles S_1 and S_2 have two common tangents. In that case, we can take them by an axial inversion to two points S_1' and S_2'; the required circle Σ will go over to a circle Σ' passing through the points S_1' and S_2' and tangent to the known line l'. This circle is easy to construct (see, for example, the solution of problem 36(a)); the problem can have up to two solutions.

Since the directions of S_1, S_2, and l can be chosen in different ways, the problem, if its solution is determined, can have up to eight solutions (see the solution of problem 60).

(b) *First solution* (close to the solution of problem 26(a) in Section 2). We suppose the circles S_1, S_2, and S_3 directed and consider separately a number of cases.

1°. The circles S_1 and S_2 have no common tangents. In that case, we can take them by an axial inversion to circles S_1' and S_2' which differ only by direction. If the image of S_3 is S_3', then the image of the required circle Σ is the point Σ' of intersection of S_1' and S_2' (see the solution of problem (a)). The problem has up to two solutions.

2°. The circles S_1 and S_2 have one common tangent (are tangent to one another). In that case, we can take S_1 and S_2 by an axial inversion to a point S_1' and a circle S_2' passing through it; suppose that here S_3 goes over to S_3'. The required circle Σ goes over to a circle Σ' tangent to S_2' at the point S_1' and tangent to the circle S_3'. Σ' is easily constructed. [Its center lies on the line OS_1', where O is the center of S_1'; it is equidistant from the center $\overline{O}$ of the circle S_3' and from a point A on the line OS_1' such that $S_1'A$ is equal to the radius of S_3'.]

3°. The circles S_1 and S_2 have two common tangents. In that case, we can take them by an axial inversion to two points S_1' and S_2'; let S_3 go over to a circle S_3'. The required circle Σ will go over to a circle Σ' passing through the points S_1' and S_2' and tangent to the circle S_3' (see problem 36(a)). The problem has up to two solutions.

Since the directions of S_1, S_2, and S_3 can be chosen in different ways, the problem has up to eight solutions.

Second solution (close to the solution of problem 26(a)). Let Σ be the required circle and a_1, a_2, and a_3 the common tangents of Σ and S_1, Σ and S_2, and Σ and S_3; all circles and lines are supposed directed (Figure 220). Clearly, the lines a_1 and a_2 intersect at the point M_1 of the radical axis o_1 of the circles S_1 and S_2 (for the tangents drawn from M_1 to S_1 and S_2 are equal, because they are tangents drawn from M to Σ), and the axial inversion with axis o_1, which takes S_1 to S_2, takes a_1 to a_2. In much the same way, the axial inversion whose axis is the radical axis o_2 of the circles S_2 and S_3, which takes S_2 to S_3, takes a_2 to a_3; the axial inversion whose axis is the radical axis o_3 of the circles S_3 and S_1, which takes S_3 to S_1, takes a_3 to a_1. Thus the sequence of three axial inversions takes the line a_1 to itself. We can obtain the line a_1 in a manner similar to the solution of problem 72; after finding a_1 we can easily construct the required circle Σ. The problem of obtaining a_1 can have up to two solutions; since the directions of S_1, S_2, and S_3 can be chosen in different ways, the problem has up to eight solutions.

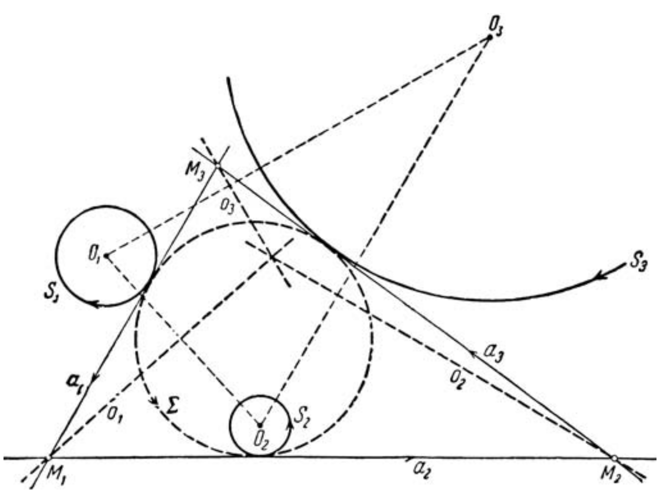


FIGURE 220

75. (a) We apply an axial inversion which takes the circles S_1 and S_2 to circles S_1' and S_2' which differ only by direction (see Theorem 2 on p. 120); let the image of S_3 be the circle (or point!) S_3'. The required circle Σ goes over to a circle Σ' such that the tangent distances of Σ' and S_1', Σ' and S_2', and Σ' and S_3' are a, b, and c. Let d be the distance between the centers of Σ' and S_1', r the (unknown) radius of Σ', and r_1 the radius of S_1' (as well as of S_2'). The relation $(*)$ on p. 90 implies that

$$d^2 - (r - r_1)^2 = a^2, \quad d^2 - (r + r_1)^2 = b^2,$$

whence

$$(r + r_1)^2 - (r - r_1)^2 = 4rr_1 = a^2 - b^2;$$

this relation enables us to determine the radius r of the circle Σ.

Now we apply a contraction by r. Let the images of the circles S_1', S_2', and S_3' be the circles S_1'', S_2'', and S_3'', and let the image of the circle Σ' be a point Σ''. Since we are given the tangent distances a, b, and c of the point Σ'' and the three known circles S_1'', S_2'', and S_3'', it is easy to find Σ'' by using the fact that the locus of points such that their tangent distance from a circle S is constant is a circle $\overline{S}$ concentric with S (to construct $\overline{S}$ we need only lay off on some tangent to S a segment of given length); then we construct the circle Σ', and, finally, the circle Σ. The problem of obtaining Σ can have two solutions, one solution, no solution, or be undetermined. Since the directions of S_1, S_2, and S_3 can be chosen in different ways, the problem has up to eight solutions.

Remark. If we are to use an axial inversion in the solution of the problem, then S_1 must not be in the interior of S_2. We leave it to the reader to look into this issue.

(b) This problem can be easily reduced to the previous one. In fact, let a circle Σ with center O intersect the known circle S_1 with center O_1 at an angle α_1 and let N denote a point of intersection of S_1 and Σ (Figure 221). The tangent NM to the circle Σ at the point N forms with the radius O_1N

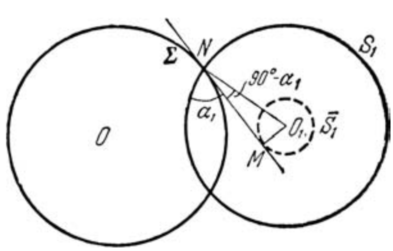


FIGURE 221

of the circle S_1 a known angle $90° - \alpha_1$; the distance O_1M of the point O_1 from this tangent is also known ($O_1M = r_1 \cos\alpha_1$, where r_1 is the radius of S_1). We draw the circle $\overline{S}_1$ with center O_1 and radius $O_1M = r_1 \cos\alpha_1$. Then the segment of the common tangent of $\overline{S}_1$ and Σ is $r_1 \sin\alpha_1$, that is, the tangent distance of the circle Σ and the circle $\overline{S}_1$ (which can be constructed if we know the circle S_1 and the angle α_1) is equal to $r_1 \sin\alpha_1$.

Thus we see that the circle Σ, which intersects the three given circles S_1, S_2, and S_3 with radii r_1, r_2, and r_3 at given angles α_1, α_2, and α_3, has known tangent distances with three other known circles $\overline{S}_1, \overline{S}_2$, and $\overline{S}_3$ (concentric with the circles S_1, S_2, and S_3); moreover, if S_1 lies inside S_2 and $90° > \alpha > \beta$, then $\overline{S}_1$ lies inside $\overline{S}_2$, and the solution of problem (b) is reduced to the solution of problem (a).

76. The circles S_1 and S_2 can be taken by an axial inversion to circles S_1' and S_2' which differ only by direction (see Theorem 2 on p. 120; the circles S_1, S_2, S_3, and S_4 are supposed directed). Let the images of the circles S_3 and S_4 be circles S_3' and S_4', and let the image of the required circle Σ be a new circle (or point) Σ' (Figure 222). The fact that the tangent distances of Σ' and each of the circles S_1' and S_2', which differ only by direction, are equal, implies that Σ' is a point (a "circle of zero radius"; see the relation (*) on p. 90). Since the tangent distances of this point and each of the circles S_1' and S_2' are equal, Σ' is on the radical axis r_1 of the circles S_1' and S_2' (see Section 3); in much the same way we show that Σ' is on the radical axis r_2 of the circles S_2' and S_4'. Hence Σ' is the radical center of S_1, S_2, S_3, and S_4 (see p. 53). After finding Σ' we can easily construct the circle Σ. This problem has a unique solution; if we suppose the circles S_1, S_2, S_3, and S_4 nondirected, then, in view of the freedom of choice of their directions, the problem can have up to eight solutions.

Remark. We can use an axial inversion to solve this problem even if the circle S_1 is not in the interior of S_2 (see XVI$'$ on p. 128).

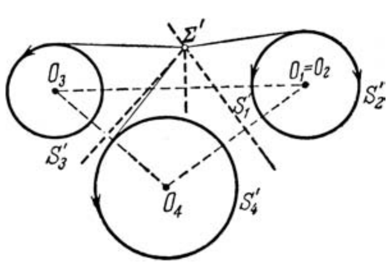


FIGURE 222

Notes to Section 5

[1] The directed tangent circles S_1 and Σ cannot have the same radii (in both magnitude and sign)—if they did, they would coincide. Hence the dilatation which takes S_1 to the point S_1' cannot take Σ to a point.

[2] This formula is applicable here for the point S_1' is in the interior of all the circles S_2, S_3, and S_4 (this follows from the fact that one can draw a tangent from the point S_1' to, say, the circle S_2'; property C of a dilatation implies that the length of the segment of this tangent between S_1' and the point of tangency is t_{12}).

[3] When solving this problem it is natural to have proved the theorem in problem 50 without using an inversion (for proofs using inversion see the solutions of problems 50 and 62). The fact that we make use of Ptolemy's theorem does not contradict this requirement, for it is easy to prove Ptolemy's theorem without the use of inversion (see, for example, the solution of problem 62(c) on p. 77 in NML 21).

[4] To claim definitely that the circles S_1 and $\overline{S}_1$ coincide, we must show that the tangent distances of S_1 and S and of $\overline{S}_1$ and S are the same not only in magnitude but also in direction (see p. 13; otherwise one might think that $\overline{S}_1$ is tangent to AB and DE at points symmetric to the points of tangency of S_1 with these same lines with respect to the points of tangency of AB and DE with the circle S). But this is easy to prove. In fact, three successive axial inversions take AB to DE (AB goes over to BC, then to CD, and, finally, to DE). The tangent distance between S and S_2 measured on the common tangent BC is opposite to the tangent distance between S and S_1 measured on AB (see the remark on p. 13); the tangent distance between S and S_2 measured on CD has the same direction as the tangent distance between S and S_1 measured on AB; finally, the tangent distance between S and $\overline{S}_1$ measured on DE is opposite to the tangent distance between S and S_1 measured on AB (see Figure 209). This implies that the segments of the common tangent AB between S and S_1 and between S and $\overline{S}_1$ coincide not only in magnitude but also in direction, which means that $\overline{S}_1$ coincides with S_1.

[5] What follows is one of the simplest proofs of this fact. Assume that $P_2Q_2 > P_2S_1'$; then $Q_2S_2' > S_2'S_1'$ (no proof is needed if all sides of $S_1'P_2Q_2S_2'$ are equal). On P_2Q_2 and $S_2'Q_2$ we lay off segments $P_2X = P_2S_1'$ and $S_2'Y = S_2'S_1'$ (Figure 211); then $Q_2X = Q_2Y$ (for $S_1'S_2' + P_2Q_2 = S_1'P_2 + S_2'Q_2$), and the bisectors of the angles P_2, S_2', and Q_2

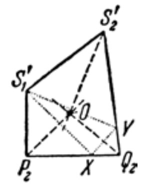


FIGURE 211

of the quadrilateral $S_1'P_2Q_2S_2'$ are perpendicular to the sides of the triangle $S_1'XY$ at their midpoints; the common point of these three angle bisectors is equidistant from all sides of the quadrilateral $S_1'P_2Q_2S_2'$, and is therefore the center of its incircle.

We note that our argument starts from Figure 210a; to make it independent of the drawing we must use the notion of directed segments (see pp. 20 and 21 in NML 8).

[6] To make our argument independent of the drawing we must use the notion of directed segments.

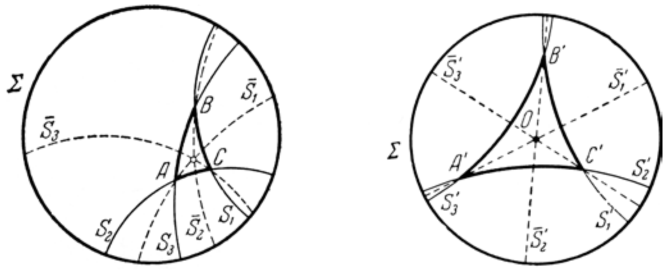
[7] See Note 6.

[8] See Note 6.

[9] If our sequence of axial inversions takes all points in the plane to points, then it is a similarity transformation (see Theorem 3 on p. 128). Since this transformation takes the circle S to itself, it is an isometry, namely, a rotation about the center O of S (in which case the problem has no solution) or a reflection in a diameter of S (in that case A_1A_n is the tangent s perpendicular to d, and the problem has two solutions). If our sequence of axial inversions takes no point in the plane to a point, then it takes all points to circles with the same radius (see the solution of problem 71); the same Theorem 3 implies that the transformation is a spiral similarity followed by a dilatation. Since this transformation must take s to itself, it is a centrally-similar rotation with center O followed by a dilatation; the problem has no solution if the angle of rotation is different from zero and is undetermined in the opposite case.

Supplement

77. Let ABC be a curvilinear triangle formed by three circles S_1, S_2, and S_3 perpendicular to the circle Σ; let $\overline{S}_1$, $\overline{S}_2$, and $\overline{S}_3$ be three other circles perpendicular to Σ, passing through the vertices of the triangle ABC and



(a) (b)

FIGURE 223

bisecting its angles; then $\overline{S}_1$, $\overline{S}_2$, and $\overline{S}_3$ meet in a point (Figure 223a; the triangle ABC is in the interior of Σ).

For proof apply a hyperbolic motion (a circular transformation that takes the interior of Σ to itself) which takes the point of intersection of the circles $\overline{S}_2$ and $\overline{S}_3$ (located in the interior of Σ) to the center O of Σ; then the circles $\overline{S}_2$ and $\overline{S}_3$ will go over to diameters $\overline{S}'_2$ and $\overline{S}'_3$ of the circle Σ. In Figure 223b the diameter $\overline{S}'_2$ forms equal angles with the circles S'_3 and S'_1, perpendicular to Σ, and the diameter $\overline{S}'_3$ forms equal angles with the circles S'_1 and S'_2, perpendicular to Σ; this implies that S'_3 coincides with the circle S''_1, obtained from S'_1 by reflection in the line $\overline{S}'_2$, and S'_2 coincides with the circle S''_1, obtained from S'_1 by reflection in the line $\overline{S}'_3$. Hence the radii of the circles S'_1, S'_2, and S'_3 are equal. But then S'_3 coincides with the circle S''_2, obtained from S'_2 by reflection in the line OA' (for among the circles perpendicular to Σ passing through a given point A' no three have the same radius; see Figure 126, p. 144). Hence OA' forms equal angles with S'_2 and S'_3, that is, OA' coincides with the circle (or line) $\overline{S}'_1$, which is the image under our hyperbolic motion of the circle $\overline{S}'_1$. The fact that $\overline{S}'_1$, $\overline{S}'_2$, and $\overline{S}'_3$ meet in a common point O implies that $\overline{S}_1$, $\overline{S}_2$, and $\overline{S}_3$ meet in a common point.

78. Let S_1, S_2, and S_3 be three circles perpendicular to a circle Σ such that S_2 and S_3 form equal angles with S_1; then, using the notations in Figure 224a, we have

$$\frac{AQ}{BQ} : \frac{AP}{BP} = \frac{AT}{CT} : \frac{AR}{CR}.$$

For proof take the point A by a hyperbolic motion to the center of the circle Σ. Then Figure 224a goes over to Figure 224b. Since the lines S'_2 and S'_3 cut the circle S'_1 at equal angles, it follows that they are symmetric with

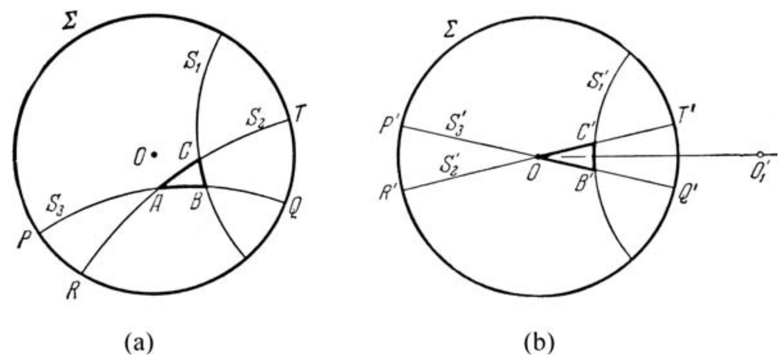


FIGURE 224

respect to the line joining O with the center O_1' of the circle S_1'. But then considerations of symmetry imply that $OB' = OC'$, so that

$$\frac{OQ'}{B'Q'} : \frac{OP'}{B'P'} = \frac{OT'}{C'T'} : \frac{OR'}{C'R'}.$$

This is so because all segments on the right side of the equality are equal to appropriate segments on its left side. It remains to use the fact that circular transformations preserve the ratio of four points.

Remark. It is not difficult to see that the line OO_1' bisects the angle $B'OC'$, is perpendicular to S_1', and bisects the arc $B'C'$ of the circle S_1'. From this it follows directly that *in an isosceles hyperbolic triangle the median, the angle bisector, and the altitude, drawn from the vertex opposite to the base, are coincident.*

79. (a) An inversion with center at the point P takes Figure 132a to Figure 225a; here we used dotted lines to draw the hyperbolic perpendiculars dropped from the points of the line $R'S'$ to the line $P'Q'$. All assertions of the problem follow directly from this figure (to show that the distances from the points of the line RS to the line PQ grow on both sides of the point B we need only note that

$$\left.\frac{A_1'K_1'}{T_1'K_1'} : \frac{A_1'L_1'}{T_1'L_1'} = \frac{\overline{A}_1'K_2'}{T_2'K_2'} : \frac{\overline{A}_1'L_2'}{T_2'L_2'} < \frac{A_2'K_2'}{T_2'K_2'} : \frac{A_2'L_2'}{T_2'L_2'}\right).$$

(b) An inversion with center at the point P takes Figure 132b to Figure 225b, which implies all the assertions of the problem (we note that

$$\left.\frac{A_1'K_1'}{T_1'K_1'} : \frac{A_1'L_1'}{T_1'L_1'} = \frac{\overline{A}_1'K_2'}{T_2'K_2'} : \frac{\overline{A}_1'L_2'}{T_2'L_2'} < \frac{A_2'K_2'}{T_2'K_2'} : \frac{A_2'L_2'}{T_2'L_2'}\right).$$

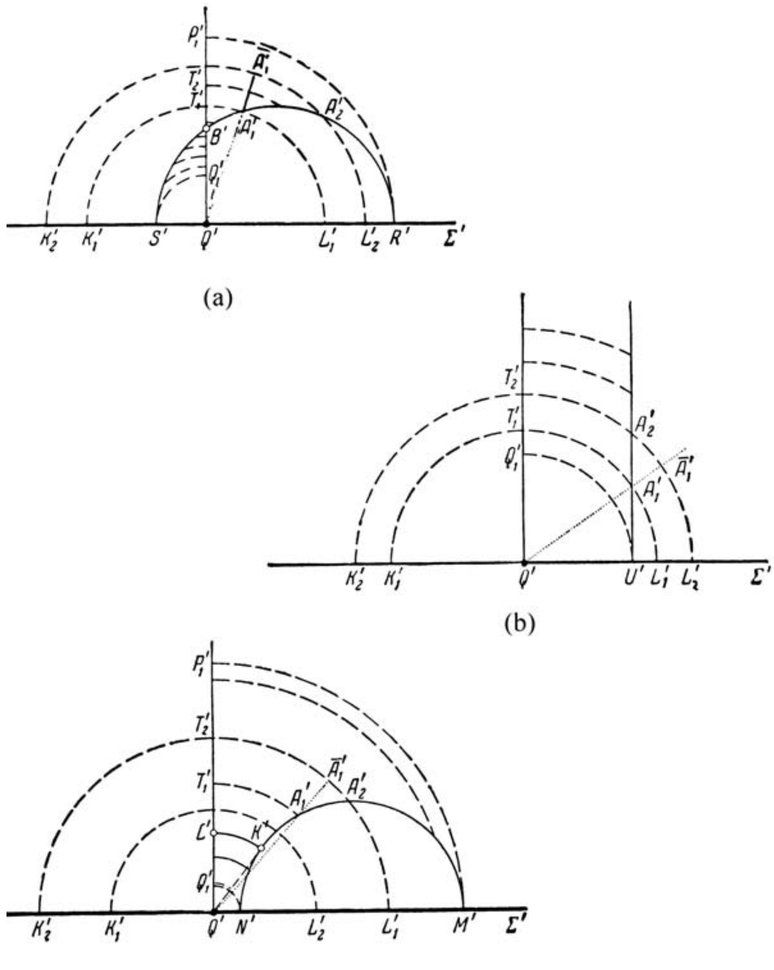


FIGURE 225

(c) An inversion with center at the point P takes Figure 132c to Figure 225c, which implies all the assertions of the problem (we note that

$$\frac{A_1'K_1'}{T_1'K_1'} : \frac{A_1'L_1'}{T_1'L_1'} = \frac{\overline{A}_1'K_2'}{T_2'K_2'} : \frac{\overline{A}_1'L_2'}{T_2'L_2'} < \frac{A_2'K_2'}{T_2'K_2'} : \frac{A_2'L_2'}{T_2'L_2'}\Bigg).$$

80. We take the point of intersection of the altitudes AK and BL of the acute-angled triangle ABC by a hyperbolic motion to the center of the

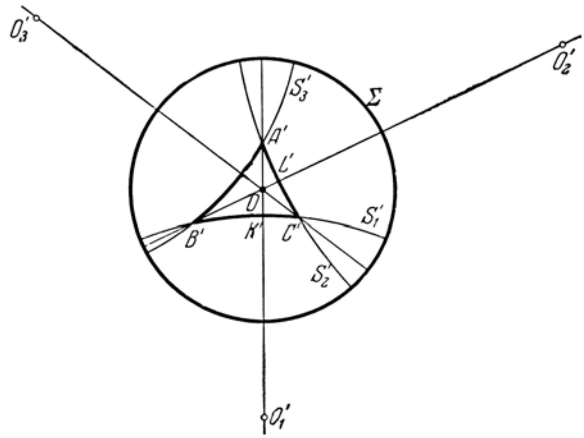


FIGURE 226

circle $\mathbb{K}$ (the altitudes of an acute-angled triangle must meet in a common point in its interior; see the beginning of the solution of problem 104 in NML 24, p. 227); this brings us to Figure 226. Let O_1', O_2', and O_3' be the centers of the circles S_1', S_2', and S_3' (which form the triangle $A'B'C'$) and let r_1, r_2, and r_3 be their radii; further, let $O_1'O_2' = d_3$, $O_1'O_3' = d_2$, and $O_2'O_3' = d_1$. The center O of the circle Σ is the radical center of the circles S_1', S_2', and S_3' perpendicular to Σ (see pp. 55 and 53–53). The line OA' is the radical axis of S_2' and S_3' (their common chord); since it is perpendicular to S_1', O_1' lies on that line. Thus O_1' has equal powers with respect to S_2' and S_3'; hence $d_3^2 - r_2^2 = d_2^2 - r_3^2$, or $d_2^2 + r_2^2 = d_3^2 + r_3^2$. In much the same way we show that $d_1^2 + r_1^2 = d_3^2 + r_3^2$. Hence

$$d_1^2 + r_1^2 = d_2^2 + r_2^2, \quad d_1^2 - r_2^2 = d_2^2 - r_1^2,$$

that is, the center O_3' of the circle S_3' has the same powers with respect to S_1' and S_2', and thus lies on their radical axis OC'; this implies that OC' is perpendicular to S_3'. Hence the lines OA', OB', and OC' are the (hyperbolic) altitudes of the triangle $A'B'C'$, that is, the three altitudes meet in a common point; this implies that the altitudes of the initial triangle ABC also meet in a common point.

If the triangle ABC is obtuse but the altitudes AK and BL meet in a common point, then the proof remains valid; that is, in that case all three altitudes meet in a common point. But it can happen that the altitudes AK and BL have no point in common; then the theorem fails (see the solution of problem 104 in NML 24, p. 227).

81. We use a hyperbolic motion to move the vertex A of an arbitrary triangle ABC to the center O of the circle Σ; then the triangle ABC goes over to the triangle $OB'C'$ shown in Figure 227. Clearly, the sum of the angles of the triangle $OB'C'$ (which is equal to the sum of the angles of the triangle ABC) is smaller than the sum of the angles of the rectilinear triangle with the same vertices, that is, it is less than 180°.

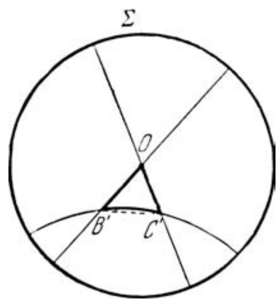
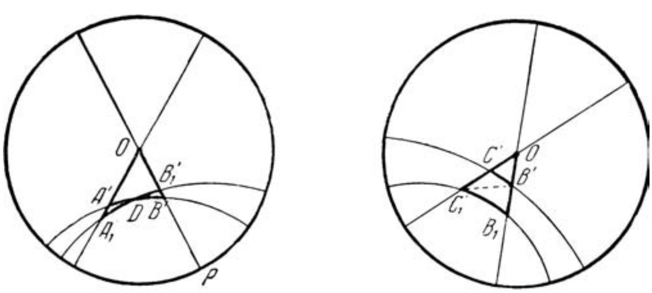


FIGURE 227

82. We take the vertex A of the first triangle ABC by a hyperbolic motion to the center O of the disk $\mathbb{K}$; we move the second triangle $A_1B_1C_1$ by a hyperbolic motion so that its vertex A_1 coincides with the same point O and its sides are on the corresponding sides of the first triangle (this is possible because the angles A and A_1 are equal). If the triangles ABC and $A_1B_1C_1$ were not congruent, then we would obtain either Figure 228a or Figure 228b. But in Figure 228a the sum of the angles of the triangle $B'B_1'D$ is greater than 180° (for $\angle DB_1'B' + \angle DB'B_1' = \angle DB_1'B' + \angle DB_1'O = 180°$), which is impossible by the result of the previous problem; in Figure



(a) (b)

FIGURE 228

228b the sum of the angles of the quadrilateral $B'C'C_1'B_1'$ is 360° (for $\angle B_1'B'C' = 180° - \angle B' = 180° - \angle B'B_1'C_1'$, $\angle C_1'C'B' = 180° - \angle C' = 180° - \angle C'C_1'B_1'$), and one of the two triangles into which the quadrilateral $B'C'C_1'B_1'$ is divided by its diagonal has an angle sum which is $\geq 180°$. This contradiction proves the congruence of our triangles.

83. Follows directly from the fact that one can circumscribe just one circle about any triangle.

84. (a) Follows from the result of problem 35 in NML 21, p. 44.

Remark. In hyperbolic geometry we have a far more general theorem, which is a consequence of the proposition in problem 7(a) in Section 1 (see also the remark in the solution of problem 89).

(b) Follows from the result of problem 3 in Section 1.

85. Follows from the result of problem 2(a) in Section 1.

Remark. Similarly, the locus of points of intersection of perpendicular cycles S_1 and S_2, tangent to a fixed cycle S at two of its given points A and B, consists of two cycles (see problem 5 in Section 1).

86. Follows from the result of problem 5 in Section 1.

87. Clearly, there is no difference between a hyperbolic pencil of cycles and a Euclidean pencil of circles (see Section 3), except that the definition of hyperbolic points forces us to consider not complete circles but only their arcs in the interior of the disk $\mathbb{K}$ (however, see the Remark in the solution of problem 89). Hence the last assertion in the problem follows directly from the results in Section 3 (see pp. 46–50).

An elliptic pencil of circles is defined by giving the points P and Q through which all circles of the pencil pass. Hence in hyperbolic geometry there are six different "elliptic pencils of cycles" corresponding to the six variants of the location of the points P and Q:

1°. P and Q are in the interior of $\mathbb{K}$;
2°. P is in the interior of $\mathbb{K}$ and Q is on the circle Σ;
3°. P is in the interior of $\mathbb{K}$ and Q is outside $\mathbb{K}$;
4°. P and Q are on the circle Σ (see Figure 133b on p. 152);
5°. P is on the circle Σ and Q is outside $\mathbb{K}$;
6°. P and Q are outside $\mathbb{K}$.

A hyperbolic pencil of cycles can be defined as a pencil perpendicular to an elliptic pencil. Hence to the six types of "elliptic pencils of cycles" there correspond six types of "hyperbolic pencils of cycles." Nevertheless, it is useful to single out one more special case, namely, the case where the

circle Σ itself belongs to the pencil under consideration (see Figure 133a on p. 151); the elliptic pencil of type 3°, perpendicular to this pencil, is characterized by the fact that the points P and Q are symmetric with respect to the circle Σ—there is a definite viewpoint which tells us that this type of "elliptic pencil of cycles" should also be regarded as special. Thus we have a total of seven types of "hyperbolic pencils of cycles."

Finally, a parabolic pencil of circles is characterized by prescribing a point A and a line l passing through A—all circles of the pencil are tangent to l at A. This yields the following six "parabolic pencils of cycles" in hyperbolic geometry:

1°. A lies inside $\mathbb{K}$;

2°. A lies on Σ and l intersects Σ (see Figure 133c on p. 153);

3°. A lies on Σ and l is tangent to Σ;

4°. A lies outside $\mathbb{K}$ and l intersects Σ;

5°. A lies outside $\mathbb{K}$ and l is tangent to Σ;

6°. A lies outside $\mathbb{K}$ and l has no points in common with Σ.

Thus we have a total of $6 + 7 + 6 = 19$ different types of pencils of cycles. We suggest that the reader should make appropriate drawings and determine the kinds of cycles in each of the 19 pencils.

88. We consider two pencils of cycles of hyperbolic geometry: the pencil $\overline{\Pi}$, consisting of all cycles perpendicular at the same time to S_1 and S_2, and the pencil Π perpendicular to $\overline{\Pi}$ (and therefore including the cycles S_1 and S_2). We assume that among the cycles of the pencil $\overline{\Pi}$ there are non-Euclidean circles; let $\overline{S}_1$ and $\overline{S}_2$ be two such circles, and let r be the line of centers of these two circles. It is always possible to take r by a hyperbolic motion to a diameter of the circle Σ; then r will be a hyperbolic line (Figure 229).

The line r is perpendicular to $\overline{S}_1$ and $\overline{S}_2$; hence it belongs to the pencil Π. This implies that all cycles of the pencil $\overline{\Pi}$ are perpendicular to r; but then the centers of all circles of the pencil $\overline{\Pi}$ lie on r.

If the cycles S_1 and S_2 intersect one another, then all cycles of the pencil Π pass through their points of intersection; hence in that case the radical axis r also passes through these two points (and therefore coincides with the common chord of S_1 and S_2).

Remark. It is possible to show that the radical axes of three cycles S_1, S_2, and S_3 taken two at a time always belong to a single pencil of hyperbolic lines. In particular, if two radical axes meet in a point Z, then the third radical axis also passes through that point. In that case, the point Z is called the radical center of the three cycles S_1, S_2, and S_3.

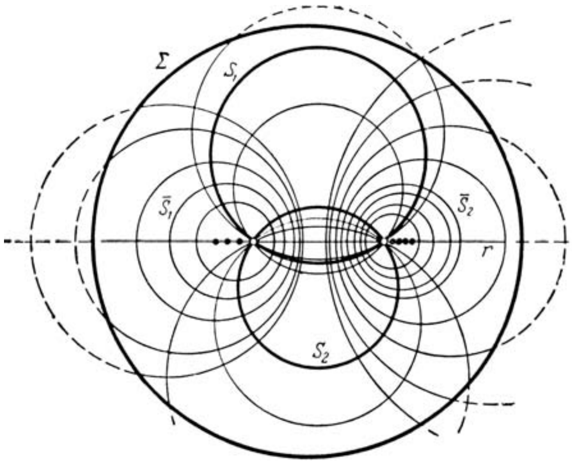


FIGURE 229

89. (a) Follows from Theorem 1 on p. 3.

(b) Follows from properties B and C of a reflection in a circle (see Section 1).

Remark. One should keep in mind that a reflection in a cycle S is not a transformation of hyperbolic geometry which takes every point to some point. For example, a reflection in the circle S in Figure 230 takes the exterior of S to the annulus between the circles S and σ (σ is symmetric to Σ with respect to S); as for the points in the interior of σ, no points are their images under our transformation (for example, the image of the equidistant S_1 under reflection in S is the arc $P'Q'$ of the circle S_1'). This fact greatly complicates all arguments involving reflections in cycles.

Our present difficulty is similar to the difficulty which we encountered in Section 5 when defining a dilatation by a negative quantity $-a$ (see p. 85)—here some points

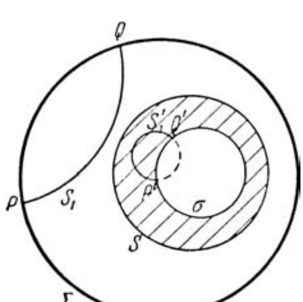


FIGURE 230

do not go over to any points and there some circles (with radii smaller than a) did not go over to any circles. This awkwardness can be eliminated in much the same way as the awkwardness in Section 5. Namely, we will regard all hyperbolic points as being double points, that is—just as in Section 5—we will speak of directed points, points with an assigned "direction of rotation," either clockwise or counterclockwise (this can be indicated by an arc with an arrow). Further, we will stipulate that all points of the disk $\mathbb{K}$ have a counterclockwise direction of rotation, and that points that differ only by their direction of rotation are symmetric with respect to the circle Σ. If, in addition, we stipulate that the points of Σ are at infinity (we do not assign to these points a direction of rotation), then the set of all points of hyperbolic geometry will coincide with the set of points of the plane (supplemented by the "infinitely distant point," directed counterclockwise, corresponding to the center of the disk $\mathbb{K}$). We can now regard the cycles of hyperbolic geometry as directed; moreover, a (directed) point A can be viewed as belonging to a directed cycle S only if the direction of rotation of A agrees with the direction of rotation associated with motion on the cycle S (see the schematic Figure 231 in which the point A belongs to the cycle S and the point B does not).

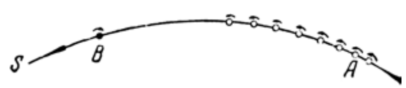


FIGURE 231

Now by an equidistant with axis PQ we mean the locus of points at a constant distance from the line PQ and located on both sides of PQ; moreover, the upper and lower branches of the equidistant must be oppositely directed (see the schematic Figure 232a); the equidistant is represented by a complete circle intersecting Σ (Figure 232b). We will include in the set of cycles lines without assigning to them

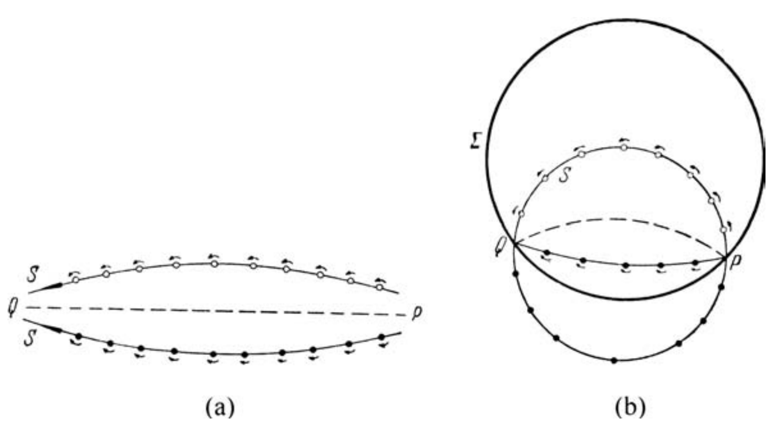


FIGURE 232

directions (this is similar to what we did in Section 5, where we assigned no direction to points[1]) and the (also undirected) "infinitely distant circle" Σ. Then the totality of cycles of hyperbolic geometry will coincide with the totality of circles and lines in the plane.

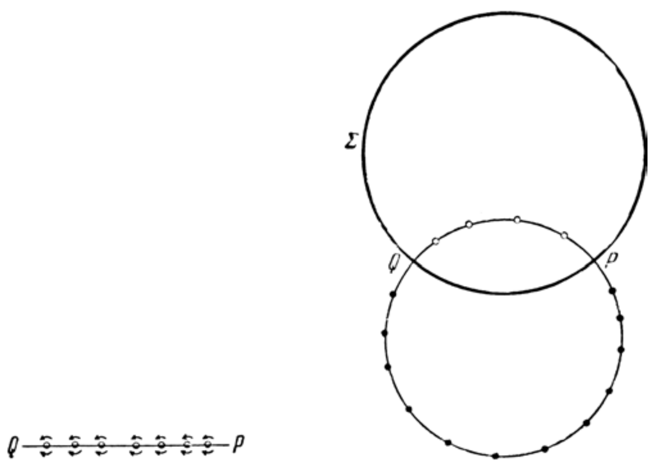
With this extension of the meaning of "point of hyperbolic geometry" and "cycle of hyperbolic geometry," a reflection in a cycle becomes a transformation which takes every (directed or infinitely distant) point to a new (directed or infinitely distant) point, and every cycle to a new cycle. It is natural to call transformations of hyperbolic geometry which take cycles to cycles circular transformations; and transformations which take lines to lines—linear transformations. Then we have the following two-part theorem, completely analogous to Theorems 1 and 2 in Section 4 (p. 73): *Every transformation of hyperbolic geometry which is circular and linear is a hyperbolic motion, and every circular transformation which is not linear can be realized by a reflection in a cycle S possibly followed by a hyperbolic motion.*[2] To prove the first of these theorems we need only note that every circular transformation which takes Σ to itself is a hyperbolic motion (see the definition of a hyperbolic motion on p. 143). Now suppose that a circular transformation K takes Σ to a cycle σ, and let the circles Σ and σ be symmetric with respect to a circle S (see the text in fine print on p. 10). Then the circular transformation K can be represented as a product of a reflection in S and some circular transformation $\overline{K}$ which takes Σ to itself (compare this argument with the proof of Theorem 2 on pp. 74–76). Now to complete the proof of the first of the stated theorems it suffices to note that a reflection in a cycle S which is not a line and not the circle Σ cannot take all lines of hyperbolic geometry to lines.

The introduction of directed points into circular transformations is convenient not only in the theory of circular transformations but also in many other questions connected with cycles. For example, in the statement of the theorem in problem 84 we need no longer require that some two of the cycles S_1, S_2, S_3, and S_4 should intersect one another in two points—this condition is automatically satisfied; also, the introduction of directed points markedly simplifies the statements of fairly complicated theorems, such as those mentioned in the Remark following the solution of problem 84.

In analogy to point circular transformations—transformations on the set of directed points which take cycles (including nondirected lines) to cycles—one could also study axial circular transformations, that is, circular transformations on the set of directed lines which take cycles (including nondirected points) to cycles. The theory of axial circular transformations of hyperbolic geometry has much in common with the theory of point circular transformations. [In this connection see the paper: I. M. Yaglom, *Projective measures on the plane and complex numbers.* Proceedings of the seminar on vector and tensor analysis, issue VII, M.-L. Gostekhizdat, 1949; this is a rather advanced paper intended for suitably qualified readers.]

Notes to Supplement

[1] We introduced the notion of directed cycles in order to establish the directions of the points belonging to such a cycle; the direction of the arrow on the arc around the point, drawn on the convex side of the cycle, must coincide with the direction of the cycle (see Figure 231). However, since a line is not convex, we cannot assign a direction to its points (see the schematic Figure 233a). It follows that we must regard lines as undirected and their points as double; this means that a line is represented by a complete circle perpendicular to the circle Σ (Figure 233b).



(a) (b)

FIGURE 233

[2] Hence in hyperbolic geometry all circular transformations are reduced, in a specific sense, to the following three fundamentally different types of transformations: (a) reflection in a circle; (b) reflection in a horocycle; (c) reflection in an equidistant.

A hyperbolic motion discussed in the text can be followed by a reversal of the direction in which we go around each point (to this transformation there corresponds a reflection in the circle Σ).

About the Author

Isaac Moisevitch Yaglom was born in 1921 in Kharkov. He graduated from Sverdlovsk University in 1942. He was professor of Geometry at Moscow State Pedagogical Institute 1957–1968. Yaglom was a mathematical intellectual equally conversant in philosophy, history and science. His erudition and his storytelling ability enriched much of his written work. He was a born educator, forever concerned with making important mathematics accessible to readers with a modest mathematical background. He is author of the three previous volumes on *Geometric Transformations* (volumes I, II, and III) all published in the Anneli Lax New Mathematical Library series by the Mathematical Association of America. He is also the author of: *A Simple non-Euclidean Geometry and Its Physical Basis*, and *Felix Klein and Sophus Lie*.